Almost everywhere convergence of sequences of (C,α) means of *m*-adic Fourier series of integrable functions

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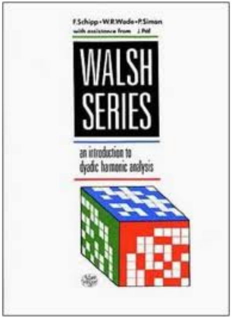
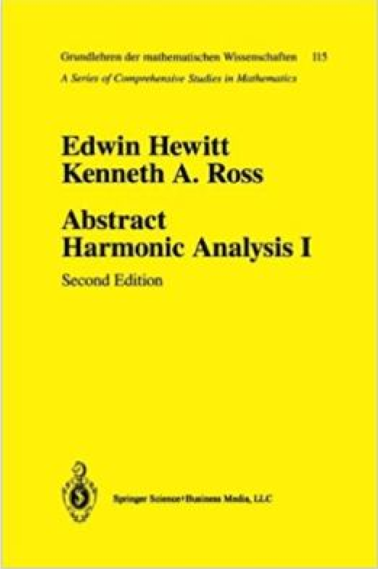
Content

Notions of Schipp-Wade-Simon-Pál



Hewitt-Ross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A Series of Comprehensive Studies in Mathematics
Edwin Hewitt
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Abstract
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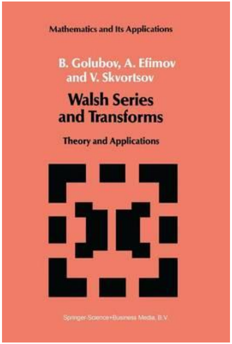
Approximation by "rectangular waves"



This work investigates convergence questions on compact totally disconnected groups of the *m*-adic integers.

Consider the Fourier series with respect to the product system of normed coordinate functions of continuous irreducible unitary representations of the coordinate groups.

It is known that the Fejér means of an integrable function on these groups converge a.e. to the function.

In this work we prove the above for (C,α) Cesàro means.

Construction of the group G_m

Let $\mathbb{P} := \mathbb{N} \setminus \{0\}$.
Let $m := (m_k, k \in \mathbb{N})$ be a sequence of positive integers with $m_k \geq 2$ $(k \in \mathbb{N})$.

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Let $\mathbb{P} := \mathbb{N} \setminus \{0\}$.
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Consider finite groups of order m_k and let us denote them by G_{m_k} $(k \in \mathbb{N})$.
The same notation for operations.
e denotes the identity $(k \in \mathbb{N})$.
Discrete topology.
μ_k: their right and left invariant Haar measure with $\mu_k(G_{m_k}) = 1$.

Construction of the group G_m

Let G_m be the compact group formed by the complete direct product of $(G_{m_k}, k \in \mathbb{N})$ with the product of topologies, operations and Haar measures:

$$G_m = \times_{k=0}^{\infty} G_{m_k}.$$

Each $x \in G_m$ is a sequence $x := (x_0, x_1, \ldots)$ with $x_k \in G_{m_k}$ $(k \in \mathbb{N})$.
If the sequence m is bounded, then the group G_m is said to be bounded.

Now **the group G_m is supposed to be bounded.**

Definitions of basic products, and that of integrals

Let $M_0 := 1,\ M_{k+1} := m_k M_k\ (k \in \mathbb{N})$.

Definitions of basic products, and that of integrals

Let $M_0 := 1, \; M_{k+1} := m_k M_k \; (k \in \mathbb{N})$.

Define the intervals on the group G_m by:

$$I_0(x) := G_m,$$
$$I_n(x) = \{y \in G_m : \; y_k = x_k \text{ for } 0 \le k < n\} \quad (x \in G_m, n \in \mathbb{P}).$$

Given $n \in \mathbb{N}$ and $x \in G_m$ then $I_n(x)$ denotes the m-adic interval of length $\frac{1}{M_n}$ which contains x.
The sets $I_n := I_n(e)$ $(n \in \mathbb{N})$ are normal subgroups of G_m.
The sets I_n form a countable neighborhood base of the product topology on G_m.

m-adic expansion

Every $n \in \mathbb{N}$ can be uniquely expressed as

$$n = \sum_{k=0}^{\infty} n_k M_k, \quad \text{with} \;\; n_k \in \{0, 1, \ldots, m_k - 1\}.$$

Structure of $\mathcal{A}_n$-measurable functions on $\mathbb{G}_m$

$\mathcal{A}$: the σ-algebra generated by the intervals $I_n(a)$ $(a \in \mathbb{G}_m, n \in \mathbb{N})$.
$G_m, \mathcal{A}$, and the restriction of the measure μ on G_m gives a probability measure space $(G_m, \mathcal{A}, \mu)$.
$\mathcal{A}_n$: the sub-σ-algebra of $\mathcal{A}$ generated by the intervals $I_n(a)$ $(a \in \mathbb{G}_m)$.
$L(\mathcal{A}_n)$: the set of $\mathcal{A}_n$-measurable functions on $\mathbb{G}_m$.
The *conditional expectation* of an $f \in L^1(\mathbb{G}_m)$ with respect to $\mathcal{A}_n$ $(n \in \mathbb{N})$ is of the form

$$(\mathcal{E}_n f)(x) := \frac{1}{\mu(I_n(x))} \int_{I_n(x)} f d\mu \quad (x \in \mathbb{G}_m).$$

m-adic maximal function

Denote by $L^p(G_m)$ the usual Lebesgue spaces and denote the L^p- norm of any function $f \in L^p(G)$ by $\|f\|_p$, $(1 \le p \le \infty)$.

The m-adic maximal function of an $f \in L^1(\mathbb{G}_m)$ is defined by

$$f^*(x) = \sup_n \left|\mathcal{E}_n f\right| = \sup_n \frac{1}{|I_n(x)|}\left|\int_{I_n(x)} f\right| \qquad (x \in G_m).$$

The m-adic Hardy space is $H(G_m) := \{f \in L^1(G_m) \mid \|f\|_H := \|f^*\|_1 < \infty\}$.

Definition of coordinate functions -in general

G: a compact group
σ: equivalence classes of continuous irreducible unitary representations.
Σ, the dual object of G: the set of all such equivalence classes σ.
d_σ: the dimension of a representation $U^{(\sigma)}$, $(\sigma \in \Sigma)$.
The **coordinate functions** for $U^{(\sigma)}$ with respect to an orthonormal base $\{\xi_1, \ldots \xi_{d_\sigma}\}$ are defined by

$$u_{i,j}^{(\sigma)}(x) := \langle U_x^{(\sigma)} \xi_i, \xi_j \rangle \quad i, j \in \{1, 2, \ldots, d_\sigma\}$$

in the representation space of $U^{(\sigma)}$.
According to the Peter-Weyl's theorem, the system $\{\sqrt{d_\sigma} u_{i,j}^{(\sigma)}, \sigma \in \Sigma, \ i, j \in \{1, \ldots, d_\sigma\}\}$ is an orthonormal base for $L^2(G)$.
If G is a finite group, then Σ is also finite.

Character system of the group G_m. The construction.

Let Σ_k denote the dual object of G_{m_k} $(k \in \mathbb{N})$.
Let $\{\varphi_k^s \mid 0 \le s < m_k\}$ be the set of all normalized coordinate functions of the group G_{m_k} and suppose that $\varphi_k^0 \equiv 1$ $(k \in \mathbb{N})$.
Thus for every $0 \le s < m_k$ there exists a representation class $\sigma \in \Sigma_k,\ i, j \in \{1, 2, \ldots, d_\sigma\}$ such that

$$\varphi_k^s = \sqrt{d_\sigma} u_{i,j}^{(\sigma)} \quad (x \in G_{m_k}, k \in \mathbb{N}).$$

Let $(\psi_n)_n$ be the product system of (φ_k^s), namely

$$\psi_n(x) := \prod_{k=0}^{\infty} \varphi_k^{n_k}(x_k) \quad (x \in G_m),$$

where n has expansion $n = \sum_{k=0}^{\infty} n_k M_k$ and $x = (x_0, x_1, \ldots)$.

A special case

If G_{m_k} is the discrete cyclic group of order m_k for each $k \in \mathbb{N}$ then the system (ψ_n) is the wellknown Vilenkin system and G_m is a Vilenkin group.

Historical background

- Gát,G., Toledo, R., Lp-norm convergence of series in compact totally disconnected groups, Analysis Math., 1997.
  G-T: The system (ψ_n) is orthonormal and complete in $L^1(G_m)$.
  G-T: L^p norm convergence: for each $f \in L^p(G_m),\ p \geq 1,\ n \in \mathbb{N}$ the partial sums S_{M_n} converge to f in L^p-norm and a.e.
  G-T: the Cesàro summability in L^p norm: if $f \in L^p(G_m),\ 1 \leq p \leq \infty$, then $\sigma_n f \to f$ in L^p-norm.
- Gát,G., Pointwise convergence of Fejér means on compact totally disconnected groups, Acta Scientiarum Mathematicarum(Szeged)60, pp 311-319, 1995.
  Gát: the pointwise convergence $\sigma_n f \to f$ a.e. $(f \in L^1(G_m))$.

m-adic Fourier coefficients, Dirichlet kernels, partial sums, Fejér means, Fejér kernel

$$\hat{f}(n) := \int_{G_m} f\overline{\psi}_n d\mu,$$

$$D_n(y,x) := \sum_{k=0}^{n-1} \psi_k(y)\overline{\psi_k(x)},$$

$$S_n := \sum_{k=0}^{n-1} \hat{f}(k)\psi_k$$

$$\sigma_0 f := 0, \qquad \sigma_n f := \frac{1}{n}\sum_{k=1}^{n} S_k f$$

$$K_n(y,x) := \frac{1}{n}\sum_{i=0}^{n} D_i(y,x)$$

$\sigma_n f = f * K_n$.

Cesàro or (C, α) kernel, Cesàro means

For given $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$ we consider numbers

$$A_k^\alpha = \frac{(\alpha+1)(\alpha+2)\cdots(\alpha+k)}{k!} \quad (\alpha \neq -k).$$

Prop: $A_n^\alpha = \sum_{k=0}^n A_{n-k}^{\alpha-1}$, $A_n^\alpha - A_{n-1}^\alpha = A_n^{\alpha-1}$, $A_n^\alpha \sim n^\alpha$.

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The *Cesàro* or (C, α) kernel for $\alpha \in \mathbb{R}$ and $n \in \mathbb{N}$:

$$K_n^\alpha(y, x) := \frac{1}{A_n^\alpha} \sum_{i=0}^n A_{n-i}^{\alpha-1} D_i(y, x) \quad (\alpha \neq -k).$$

The (C, α) Cesàro means of the integrable function f:

$$\sigma_n^\alpha f(y) := \frac{1}{A_n^\alpha} \sum_{i=0}^n A_{n-i}^{\alpha-1} S_i(y) \; (n \in \mathbb{N}, y \in G_m).$$

$$\sigma_n^\alpha f(y) = \int_{G_m} f(x) K_n^\alpha(y, x) d\mu(x) = (f * K_n^\alpha)(y) \; (n \in \mathbb{N}, y \in G_m).$$

Lemma (Gát-Toledo)

$$\sum_{s=0}^{m_k-1} \varphi_k^s(y_k)\overline{\varphi_k^s(x_k)} = \begin{cases} m_k, & \text{if } x_k = y_k \\ 0, & \text{if } x_k \neq y_k \ (k \in \mathbb{N}). \end{cases}$$

Lemma (Gát-Toledo)

If $n \in \mathbb{N}, x.y \in G_m$, *then*

$$D_n(y,x) = \sum_{k=0}^{\infty} D_{m_k}(y,x) \left(\sum_{s=0}^{n_k-1} \varphi_k^s(y_k)\overline{\varphi_k^s(x_k)} \right) \psi_{n^{(k+1)}}(y)\overline{\psi_{n^{(k+1)}}(}$$

$$D_{M_k}(x,y) = \begin{cases} M_k, & \text{if } x \in I_k(y) \\ 0, & \text{if } x \notin I_k(y). \end{cases}$$

where $(n_0, n_1, \ldots)$ *is the expansion of* n *and*

Lemma (Gát, 1997)

$$\int_{G_m \setminus I_k(u)} \sup_{n \geq M_k} |K_n(y,x)| d\mu(y) \leq C \qquad (x \in I_k(u), u \in G_m, k, n \in \mathbb{N}$$

Notations

$|n| := \max\{k \in \mathbb{N} \mid n_k \neq 0\}$.
Truncation of the smallest digits of n:
$n_{(s)} := \sum_{i=0}^{s-1} n_i M_i \quad (n, s \in \mathbb{N})$.

If $M_B \leq n < M_{B+1}$, then $|n| = B$ and $n = n_{(B+1)}$.

Lemma

$$D_{M_B - j}(y,x) = D_{M_B}(y,x) - \psi_{M_B - 1}(y)\overline{\psi_{M_B-1}}(x)\overline{D_j}(y,x).$$

Estimations for the maximal function of the Cesàro kernel

Let us introduce the following function for $n \in \mathbb{N}$ and $1 > \alpha > 0$:

$$T_n^\alpha(y,x) := \frac{1}{A_n^\alpha} \sum_{i=0}^{M_B} A_{n-i}^{\alpha-1} D_i(y,x) \quad (\alpha \neq -k).$$

Lemma

$$\int_{G_m \setminus I_k} \sup_{n \geq M_k} |T_n^\alpha(y,x)| d\mu(y) \leq C \qquad (x \in I_k(u),\ u \in G_{m_k},\ k, n \in \mathbb{N}$$

Lemma

$$\int_{G_m \setminus I_k} \sup_{n \geq M_A} |T_n^\alpha(y,x)| d\mu(y) \leq C(k-A)$$

$$(x \in I_k(u),\ u \in G_{m_k},\ k, n, A \in \mathbb{N},\ A < k).$$

Consider the maximal function:

$$K_{*,k}^{\alpha} := \sup_{n \geq M_k} |K_n^{\alpha}|.$$

Lemma

$$\int_{G_m \setminus I_k} K_{*,k}^{\alpha}(y, x) d\mu(y) \leq C \qquad (x \in I_k(u),\ u \in G_{m_k},\ k \in \mathbb{N}).$$

Lemma

$$\|K_n^{\alpha}\|_1 \leq C_{\alpha} \qquad (n \in \mathbb{N}).$$

$$\sigma_*^\alpha := \sup_{n\in\mathbb{N}} |\sigma_n^\alpha|.$$

Corollary

The maximal operator σ_*^α is of type (L^∞, L^∞).

Using Calderon-Zygmund decomposition lemma we can prove:

Lemma

σ_*^α *is of weak type (L^1, L^1).*

By standard density argument we get:

Theorem (S-Gát)

For each $f \in L^1(G_m)$ holds $\sigma_n^\alpha f \to f$ a.e.

Theorem (S.-Gát)

σ_*^α is of type (H, L^1): for each $f \in H(G_m)$ we have $\|\sigma_*^\alpha f\|_1 \leq C_\alpha \|f\|_H$.
σ_*^α is of type (L^p, L^p) $(1 < p \leq \infty)$: for each $f \in L^1(G_m)$ we have $\|\sigma_*^\alpha f\|_p \leq C_{\alpha,p} \|f\|_p$ $(1 < p \leq \infty)$.

Thank you for your attention!