ARCTIC CURVES BEYOND THE ARCTIC CIRCLE

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joint work with

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Pécs - Aug 2017

A PICTURE FROM THE BOOKLET

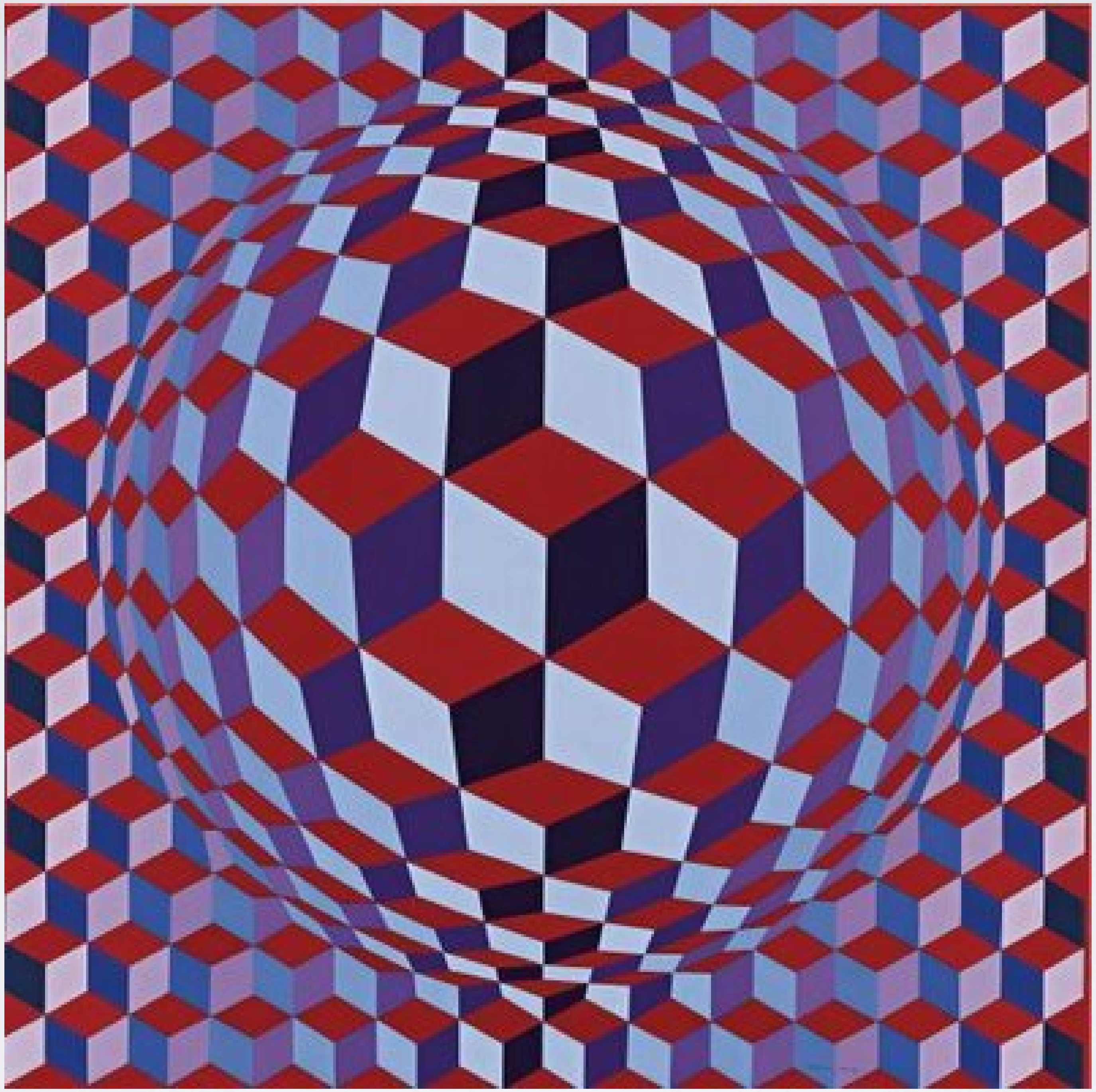
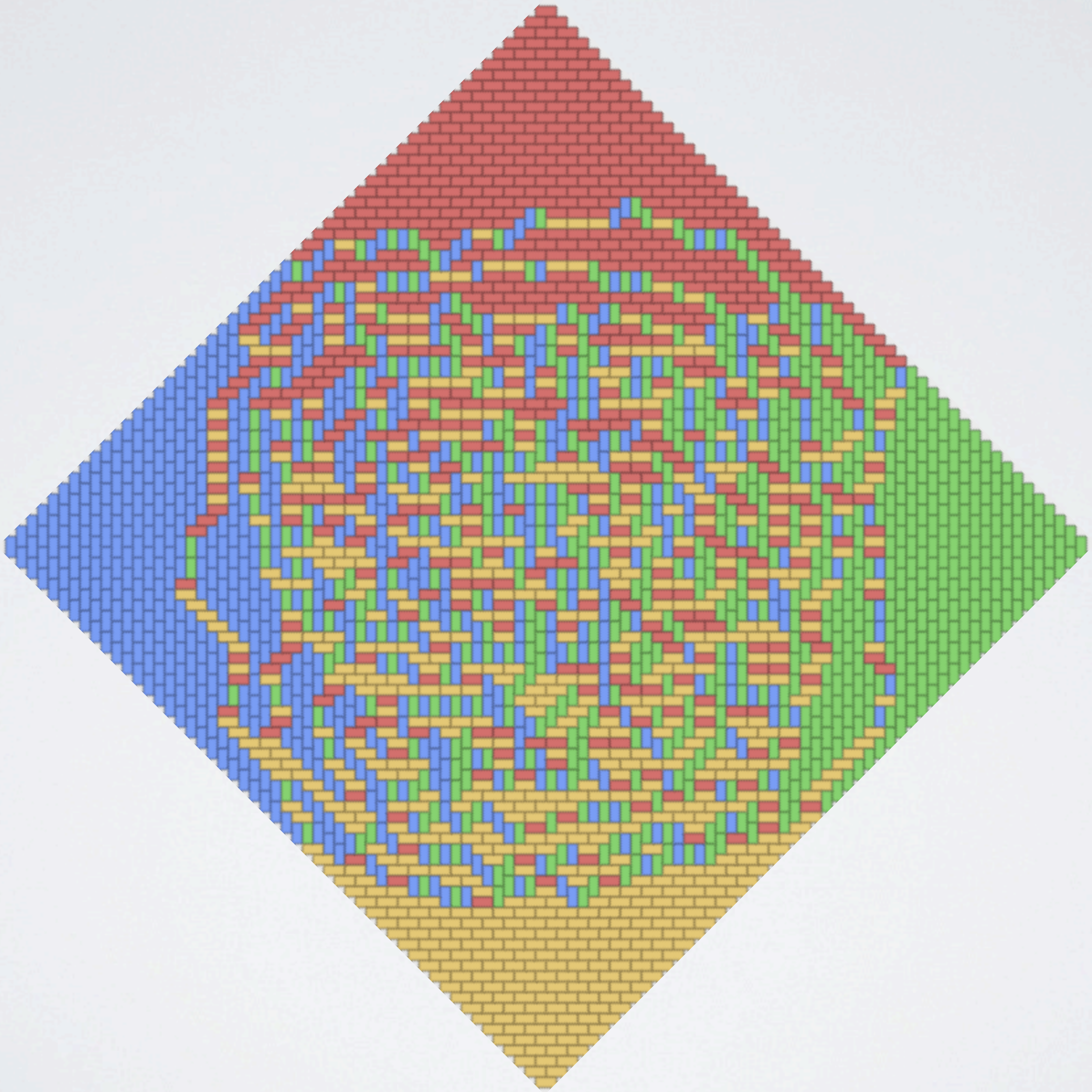


Figure 4: Picture by Vasarely

THE ARCTIC CIRCLE

Jockusch-Propp-Shor



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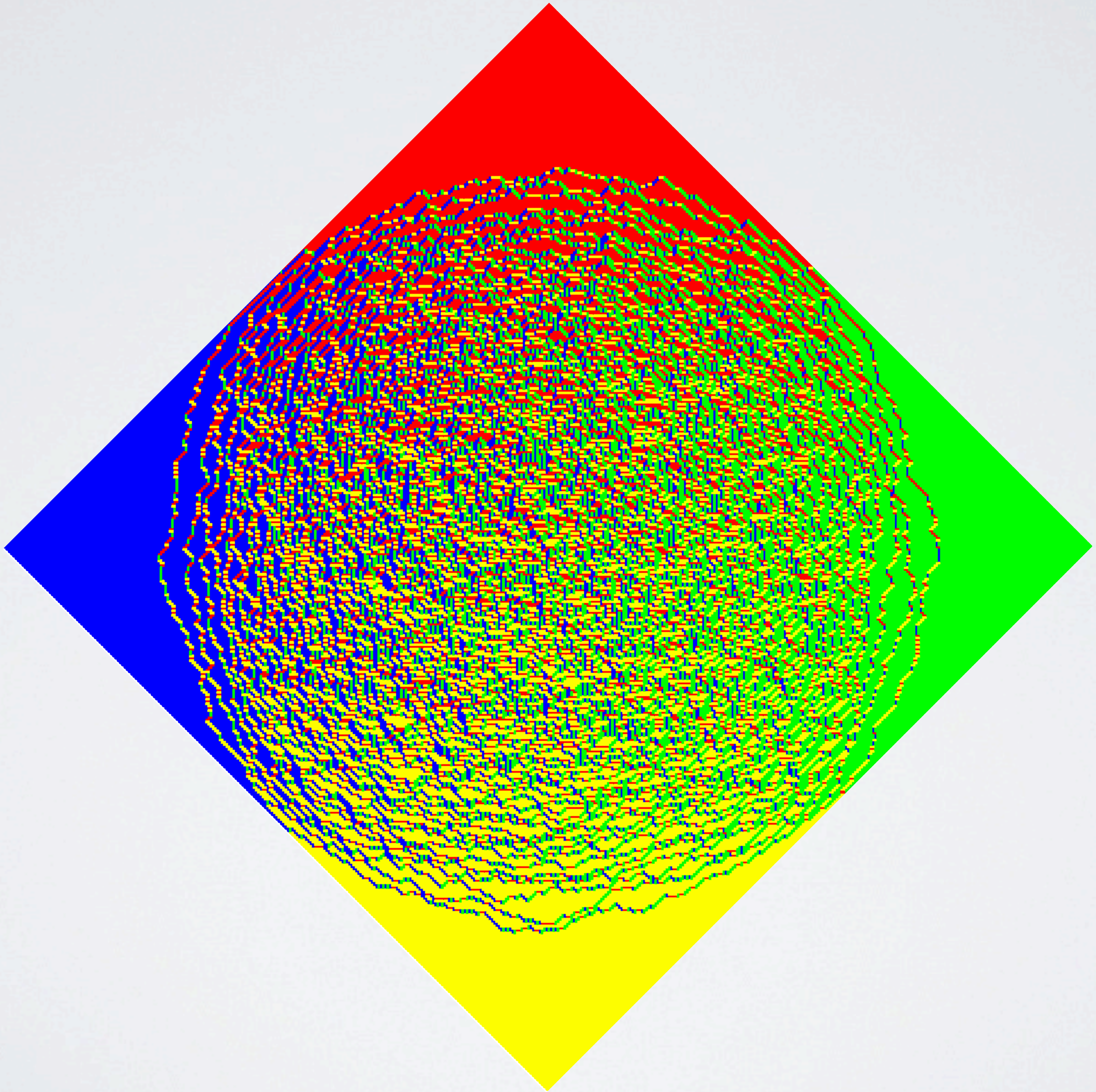

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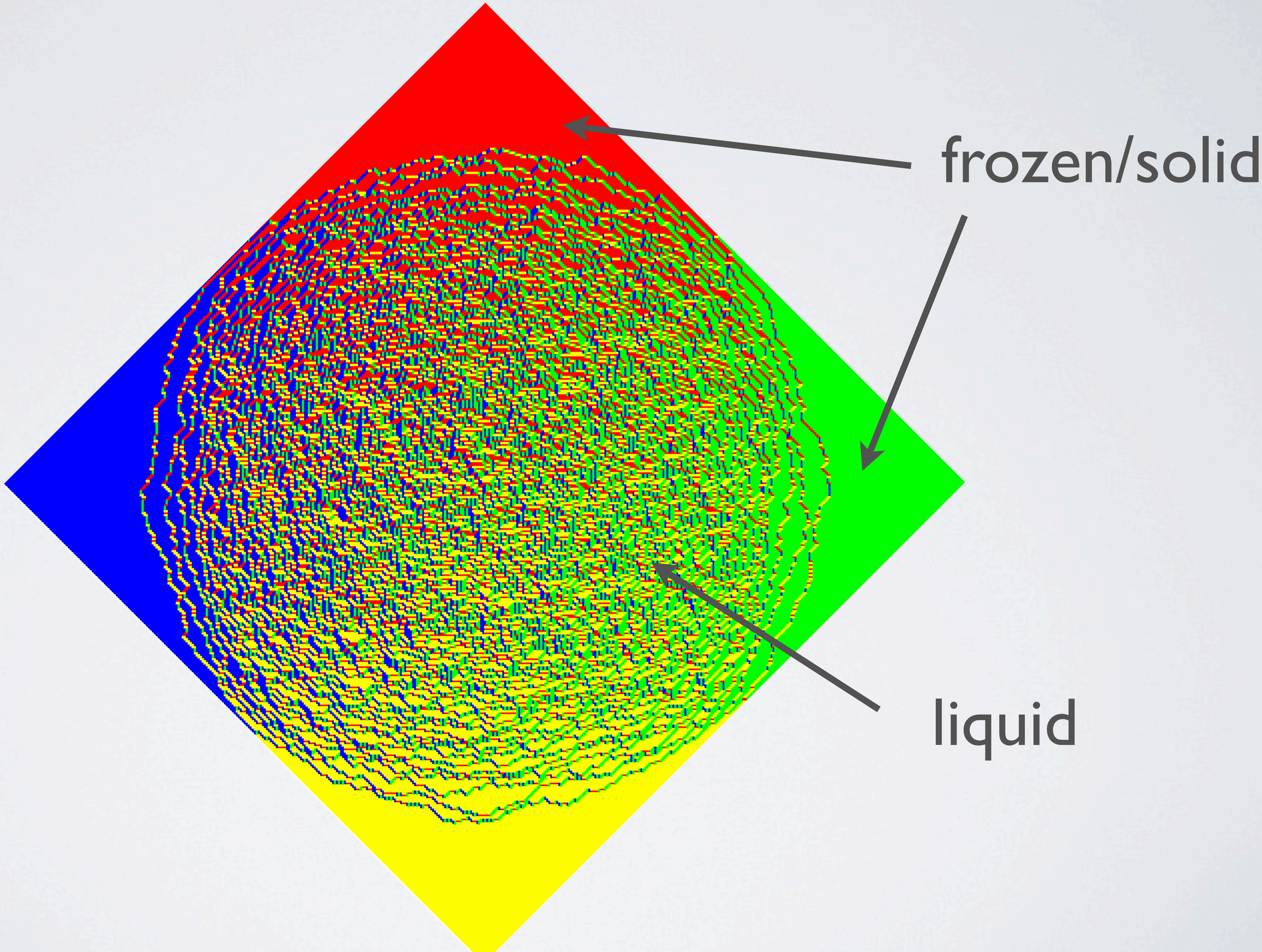
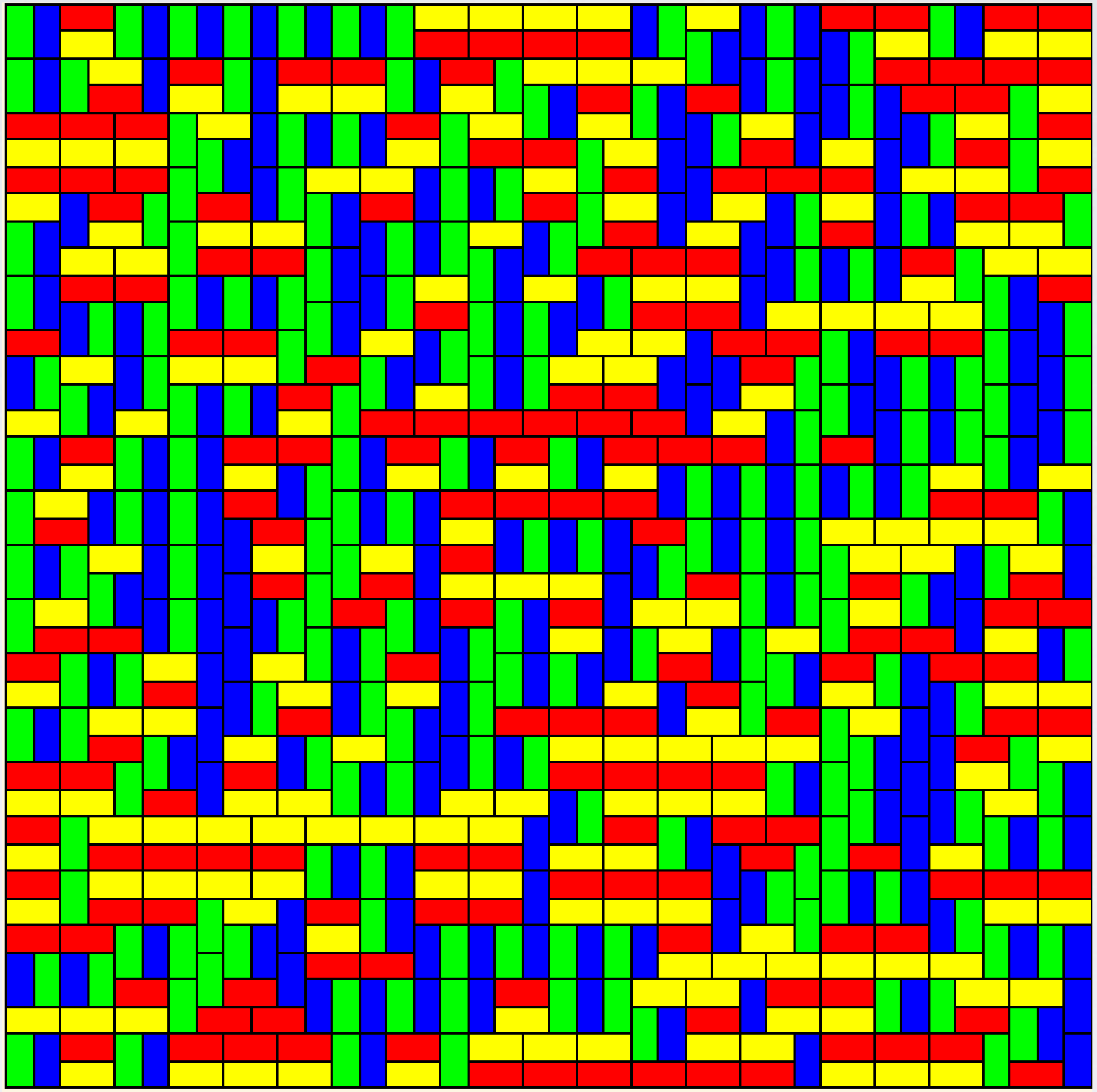
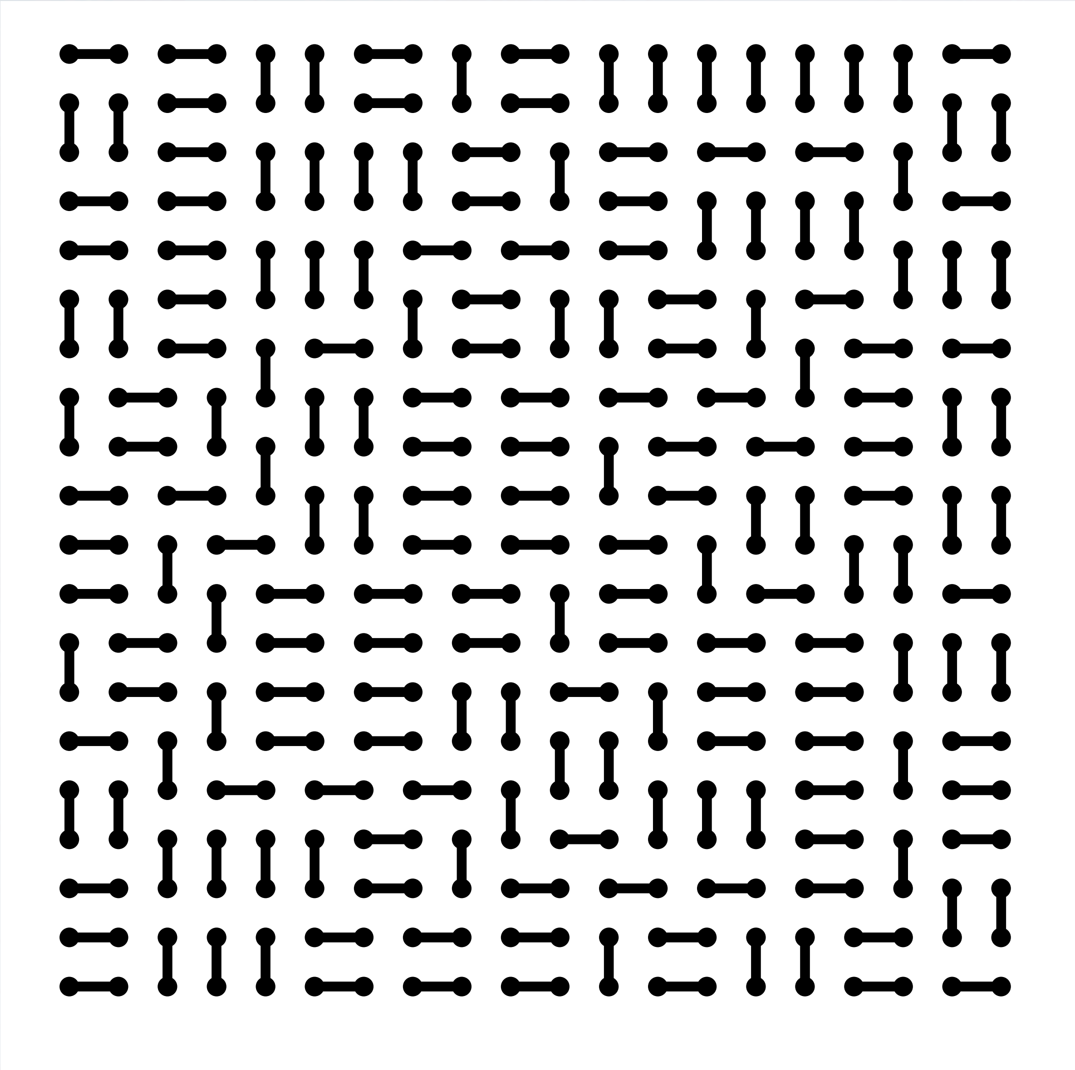


RANDOM TILING OF A SQUARE



DIMERS VS TILINGS

dimer configuration: perfect matching on a graph
(planar, doubly periodic, bipartite)



$\mathbb{Z}^2 \longleftrightarrow$ dominos

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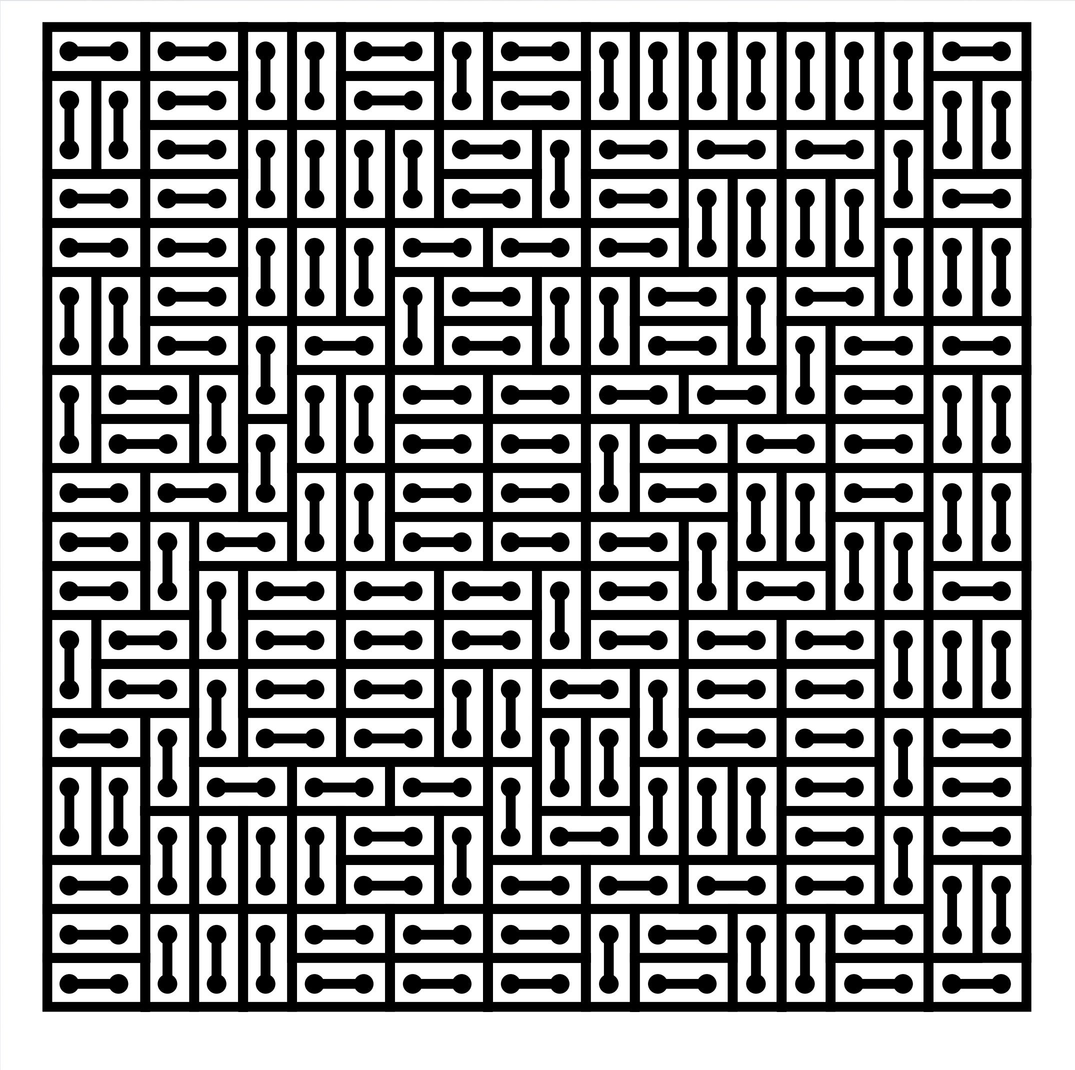

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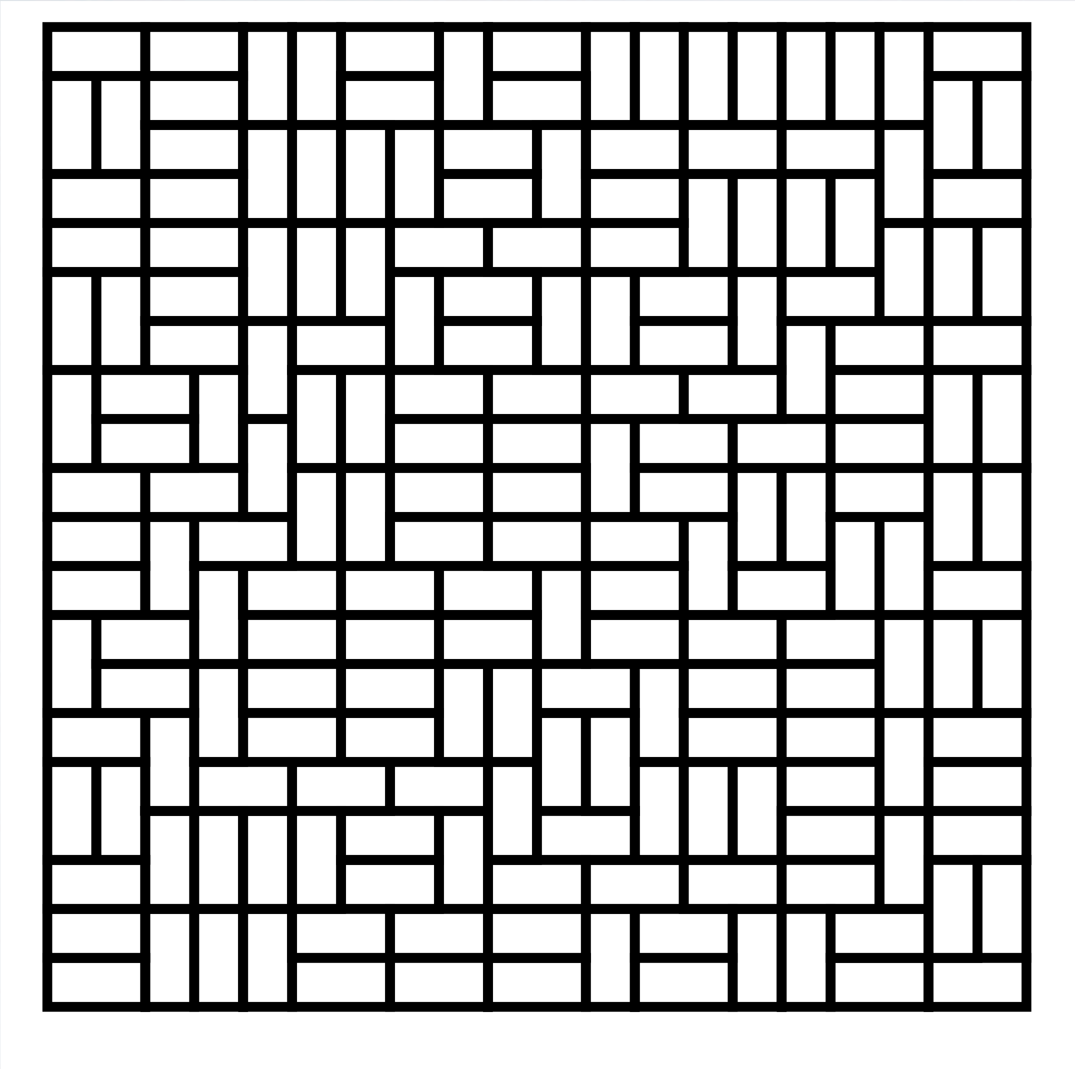

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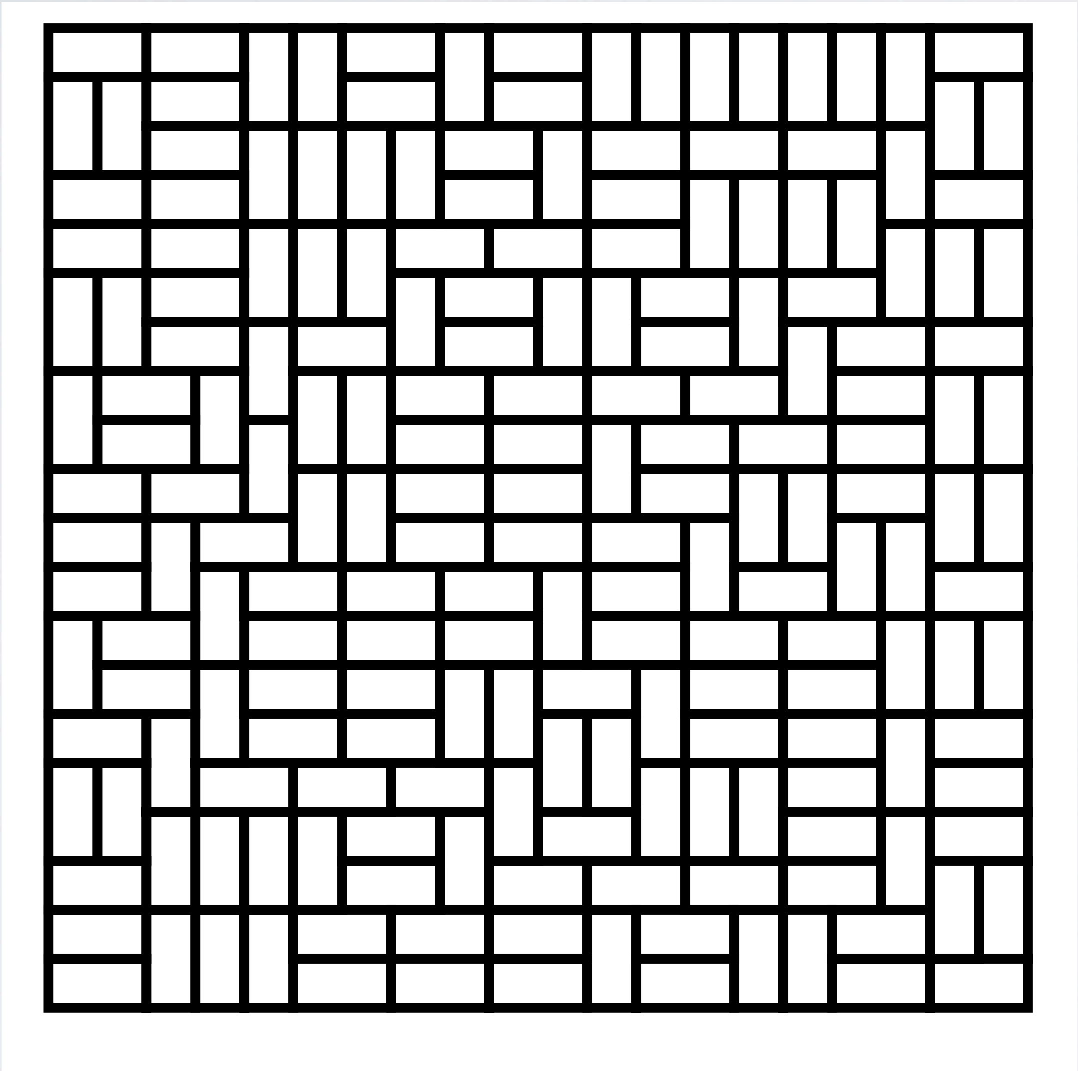

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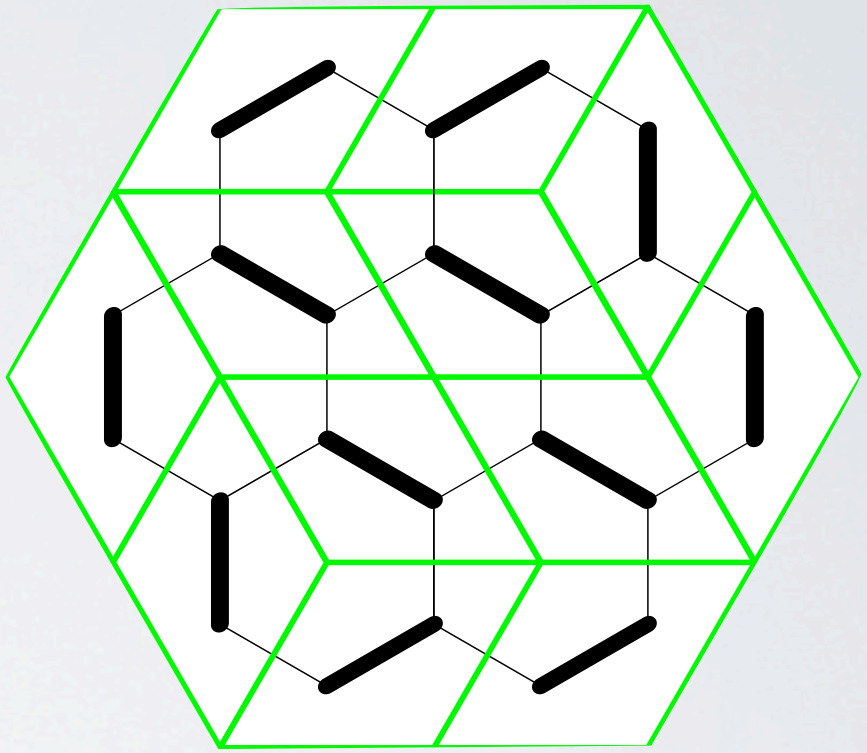
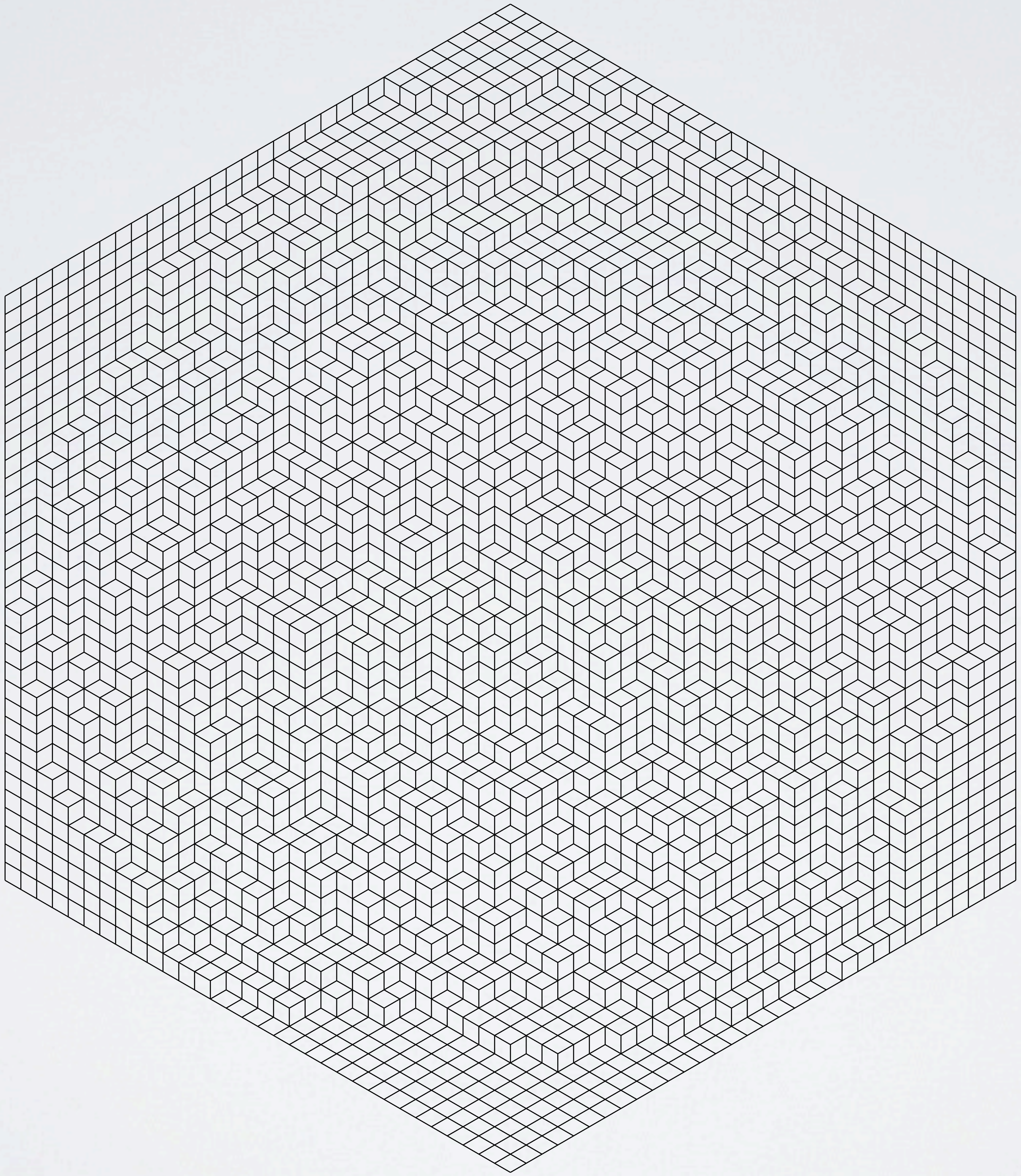


honeycomb lattice $\longleftrightarrow$ rhombus tilings

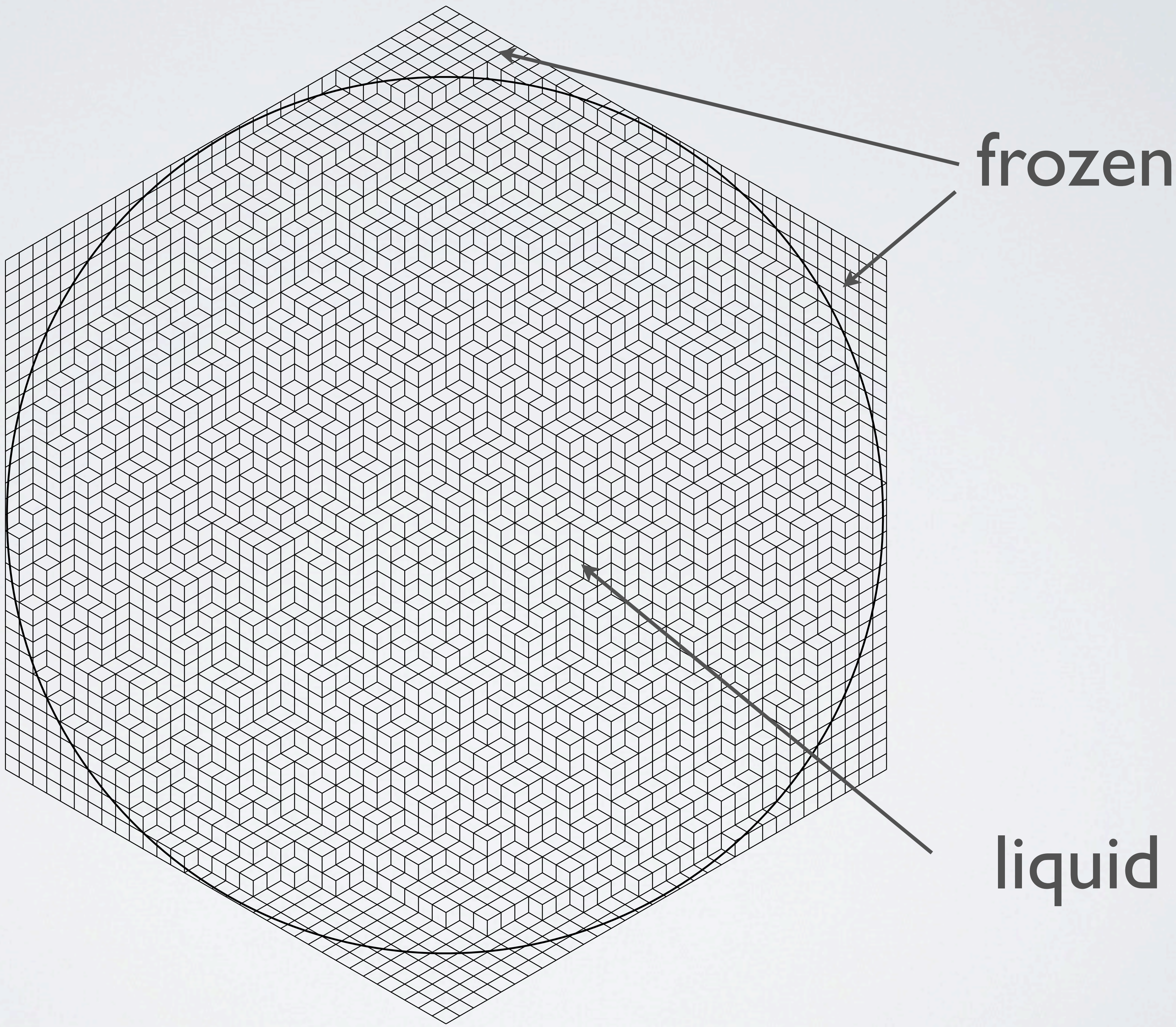
THE ARCTIC CIRCLE AGAIN

Cohn-Larsen-Propp

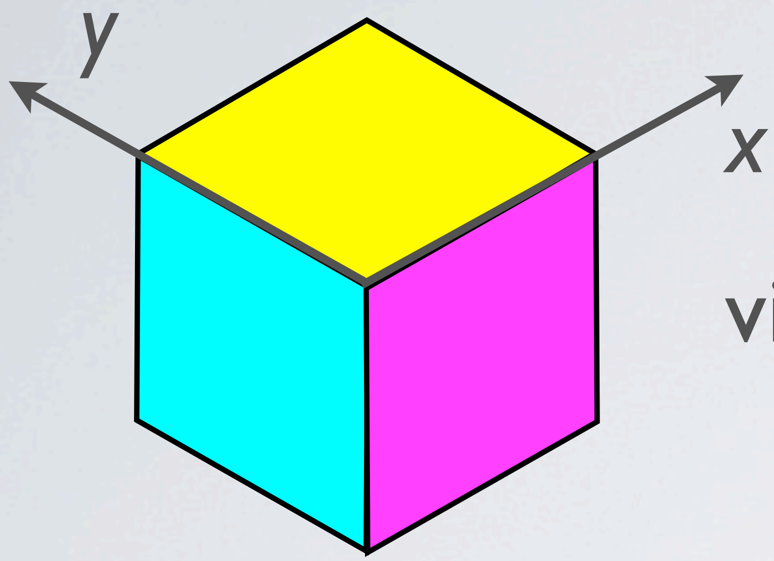


THE ARCTIC CIRCLE AGAIN

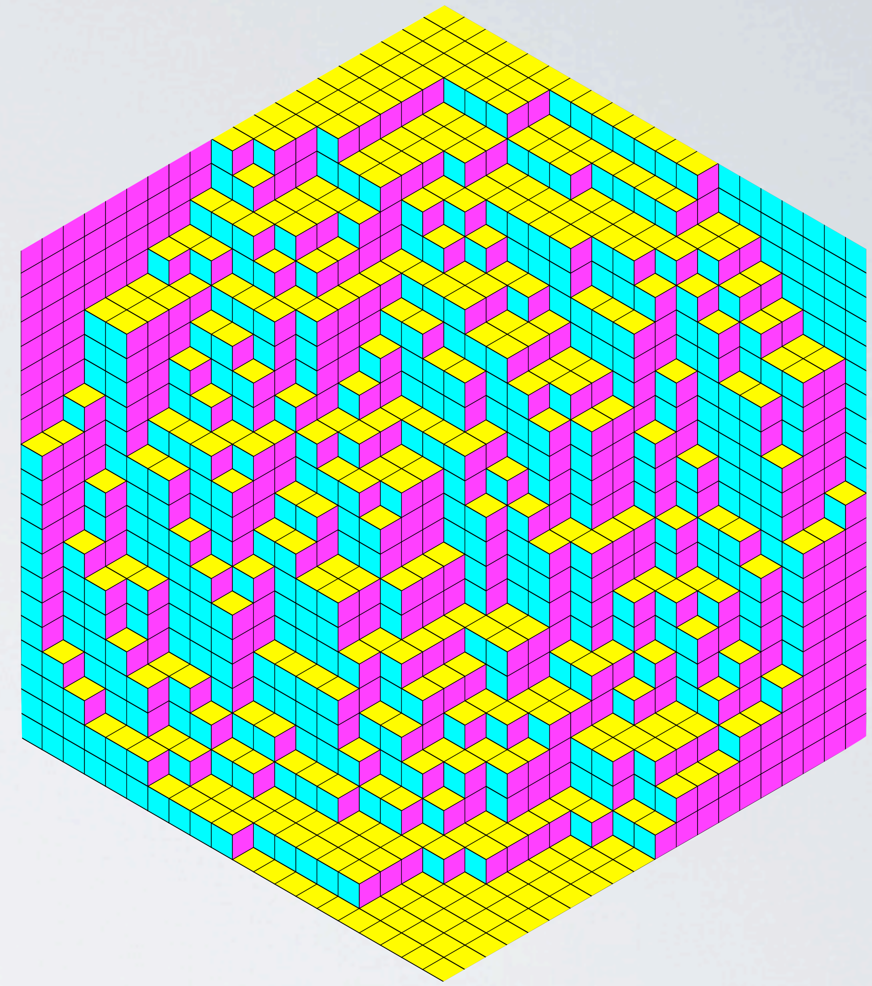
Cohn-Larsen-Propp

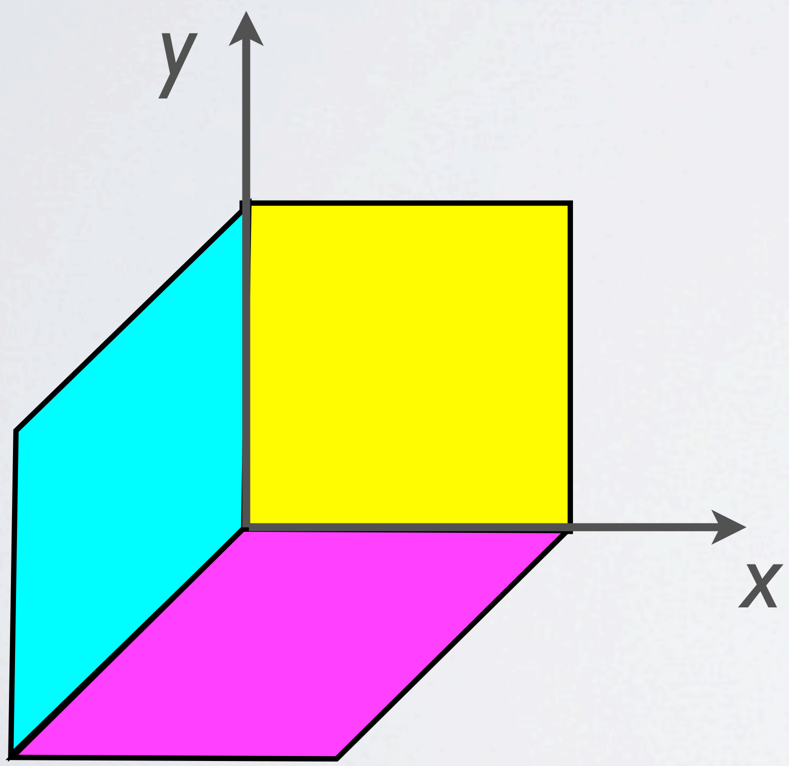


LOZENGE TILINGS



stepped surfaces
viewed from $(1,1,1)$
direction

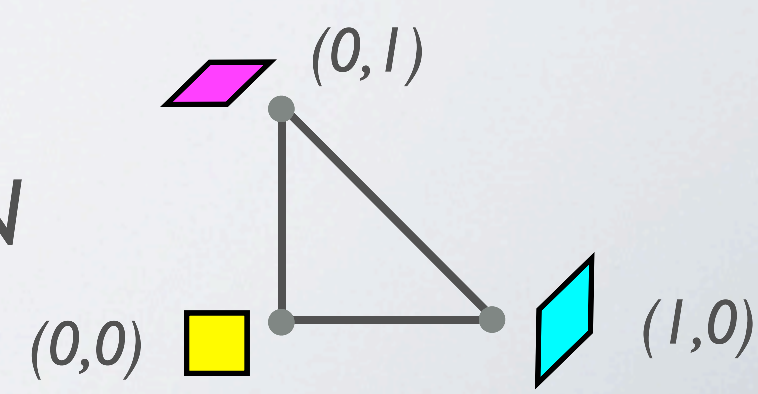




height function

"vertical height" parametrized by $(1,1,1)$
plane in Cartesian coordinates

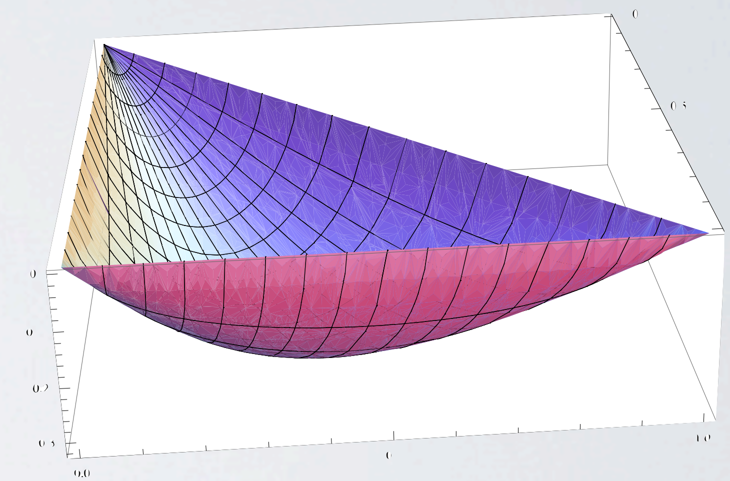
$\nabla h \in$ vertices of a triangle N



LIMIT SHAPE THEOREM

Cohn-Kenyon-Propp

For a fixed boundary height, as mesh size →0, with probability tending to 1 a random surface lies close to a deterministic surface, called *limit shape*



The limit shape is described by a *variational principle*

$h : \Omega \to \mathbb{R}$ Lipschitz

$$\min_h \int_\Omega \sigma(\nabla h), \qquad \nabla h \in N, \quad h_{|\partial\Omega} = h_0$$

analytic, strictly convex surface tension in the interior

degenerates on the boundary

SURFACE TENSION

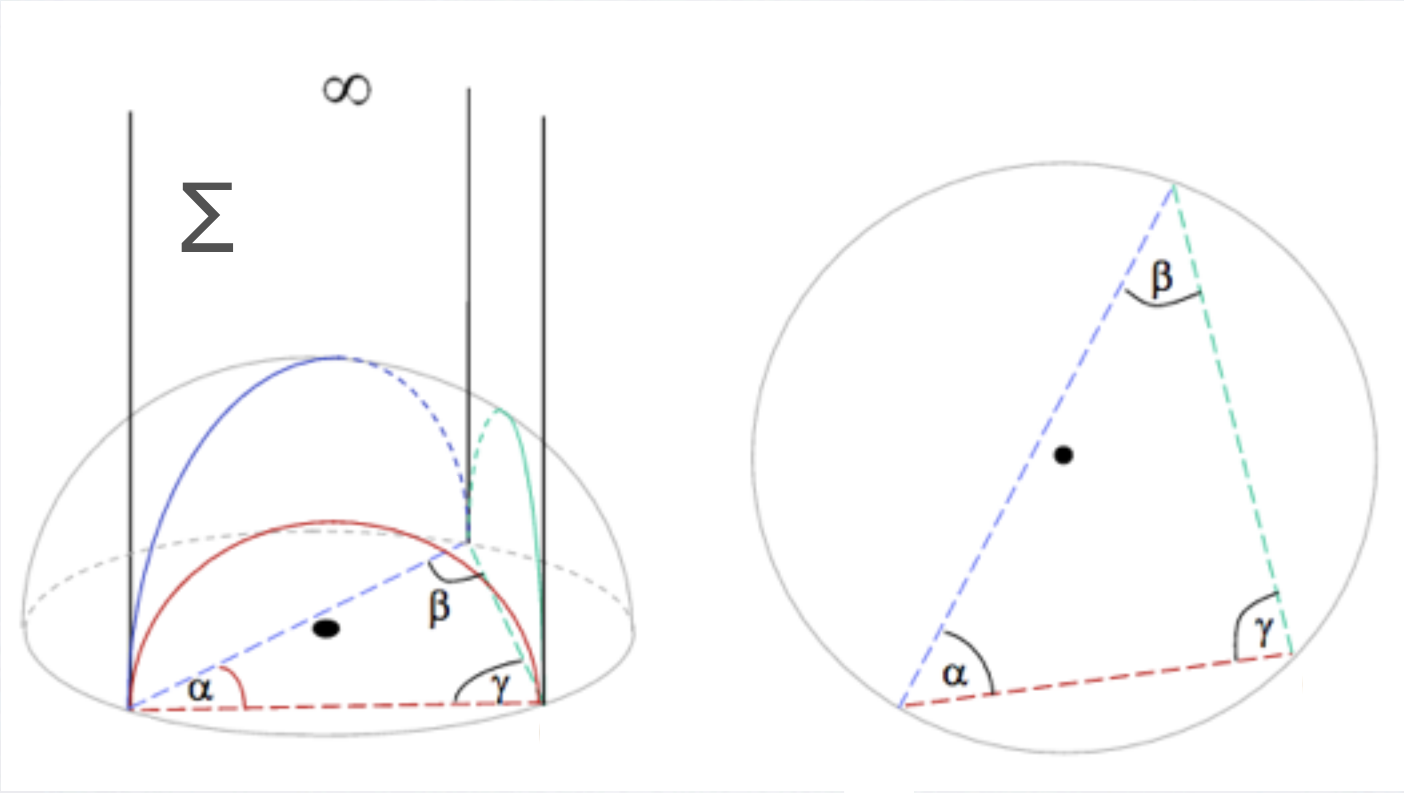
Cohn-Kenyon-Propp

Kasteleyn — dimer model "exactly solvable"

Temperly-Fisher — determinental formulae

$$\sigma(s,t) = -\frac{1}{\pi}(L(\overset{\alpha}{\pi s}) + L(\overset{\beta}{\pi t}) + L(\overset{\gamma}{\pi(1-s-t)})) = -1/\pi \operatorname{Vol}_{\mathbb{H}^3}\Sigma_{\alpha,\beta,\gamma}$$

$$L(\theta) = -\int_0^\theta \log(2\sin x)dx$$ Lobachevsky function



Limit shapes and the complex Burgers equation

by

Richard Kenyon

Brown University
Providence, RI, U.S.A.

Andrei Okounkov

Princeton University
Princeton, NJ, U.S.A.

U liquid region: where the Euler-Lagrange eq. makes sense

(h is C^1 and $\nabla h \in \operatorname{int} N$) → $h \in C^\infty(U)$

non-linear

$$\operatorname{div}(\nabla\sigma \circ \nabla h) = 0$$

Morrey

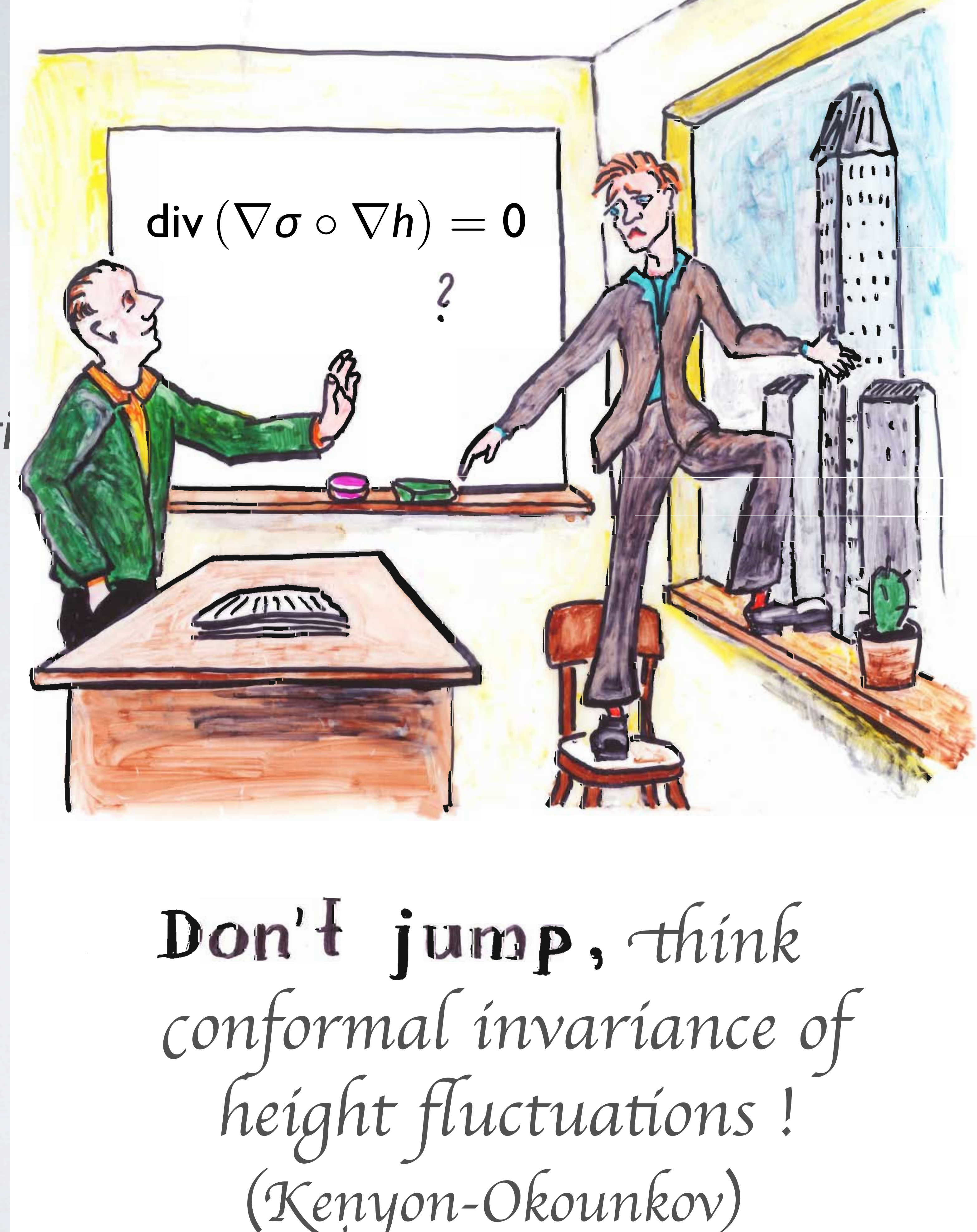
U liquid regi

sense

$h \in C^\infty(U)$

non-linear

Morrey



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U liquid region: where the Euler-Lagrange eq. makes sense

(h is C^1 and $\nabla h \in \operatorname{int} N$) $\longrightarrow$ $h \in C^\infty(U)$

Morrey

non-linear

$\rightsquigarrow$

quasilinear

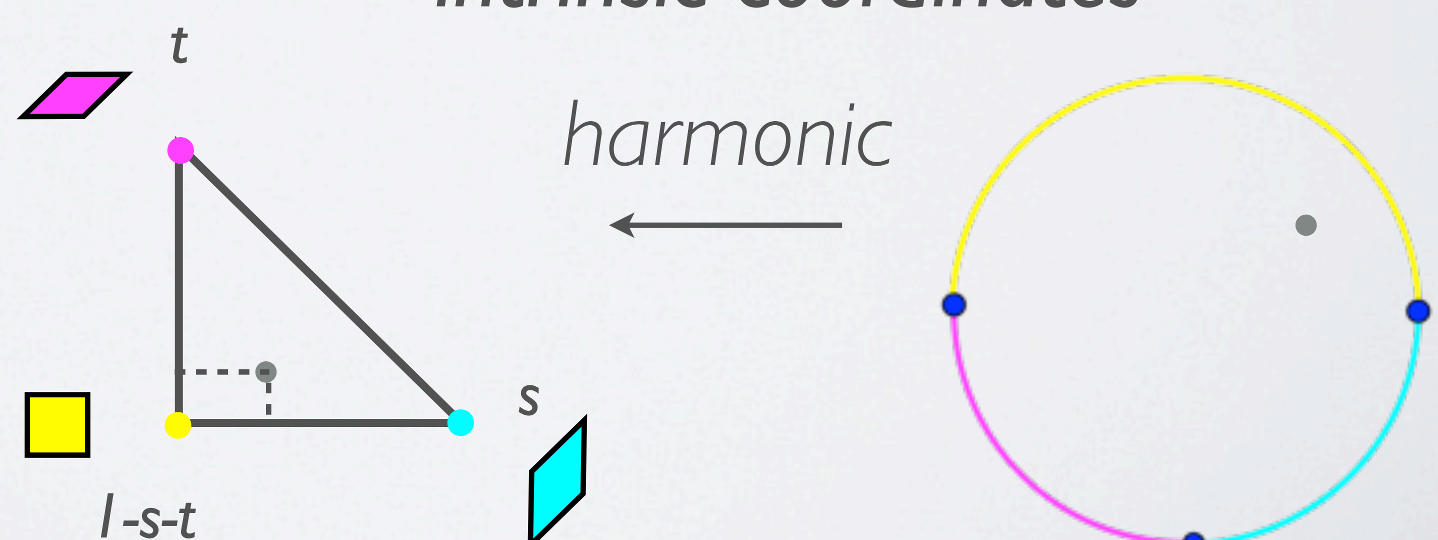
$$\operatorname{div}\left(\nabla\sigma \circ \nabla h\right) = 0$$

Kenyon-Okounkov: $(x,y) \in U \rightsquigarrow \nabla h(x,y) = (s,t) \rightsquigarrow f(x,y) \in \mathbb{D}$

"intrinsic coordinates"

$$f\colon U \to \mathbb{D}$$

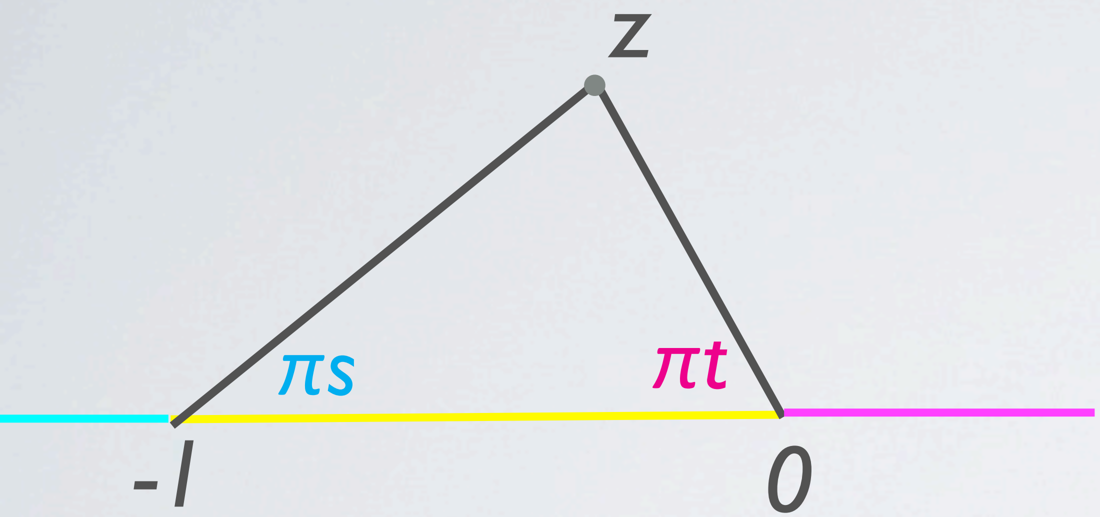
$$\bar\partial f = f\,\partial f$$



E-L IN INTRINSIC COORDINATES

$\nabla h = (s, t)$

$(s, t) \rightsquigarrow z \in \mathbb{C}_+$ 　 *z = Möbius(f)*

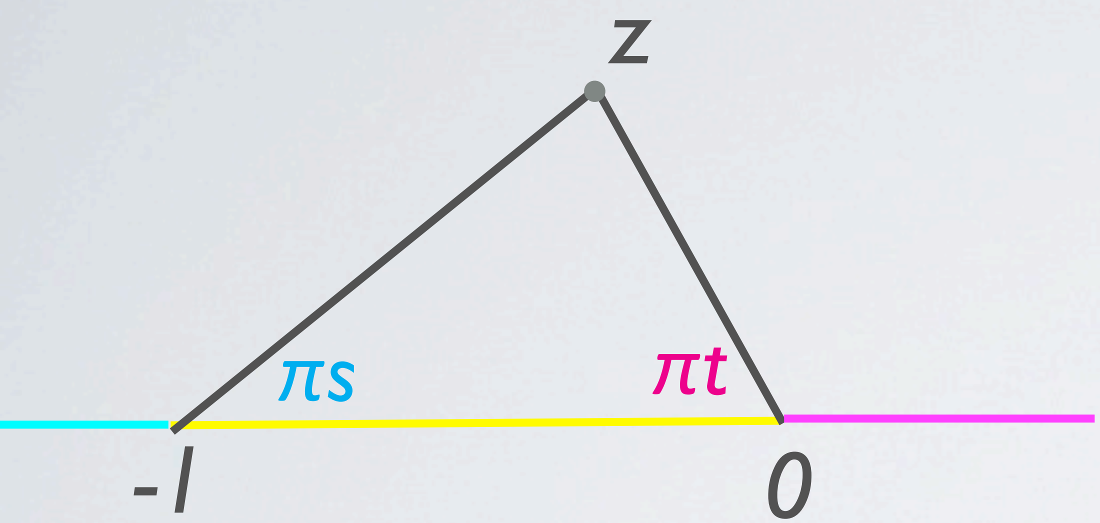


$$\sigma_s = \log \frac{\sin \pi s}{\sin \pi(1 - s - t)} = \log |z|$$

$$\sigma_t = \log \frac{\sin \pi t}{\sin \pi(1 - s - t)} = \log |1 + z|$$

E-L IN INTRINSIC COORDINATES

$$\nabla h = (s, t) \qquad (s,t) \rightsquigarrow z \in \mathbb{C}_+ \qquad z = \text{Möbius}(f)$$



$$\sigma_s = \log \frac{\sin \pi s}{\sin \pi(1 - s - t)} = \log |z|$$

$$\sigma_t = \log \frac{\sin \pi t}{\sin \pi(1 - s - t)} = \log |1 + z|$$

$$(x, y) \in U \mapsto z(x, y) \qquad \nabla\sigma \circ \nabla h = (\log |z|, \log |1 + z|)$$

$$\text{Re}\left(\frac{z_x}{z} + \frac{z_y}{1+z}\right) = \text{div}\,(\nabla\sigma \circ \nabla h) = 0$$

$$h_x = \frac{1}{\pi}\arg(-(1+z))$$

$$h_y = -\frac{1}{\pi}\arg z \qquad h_{xy} = h_{yx}$$

$$\rightarrow \quad \text{Im}\left(\frac{z_x}{z} + \frac{z_y}{1+z}\right) = 0$$

In terms of f: $\bar{\partial} f = f \partial f$

ALGEBRAIC CURVES

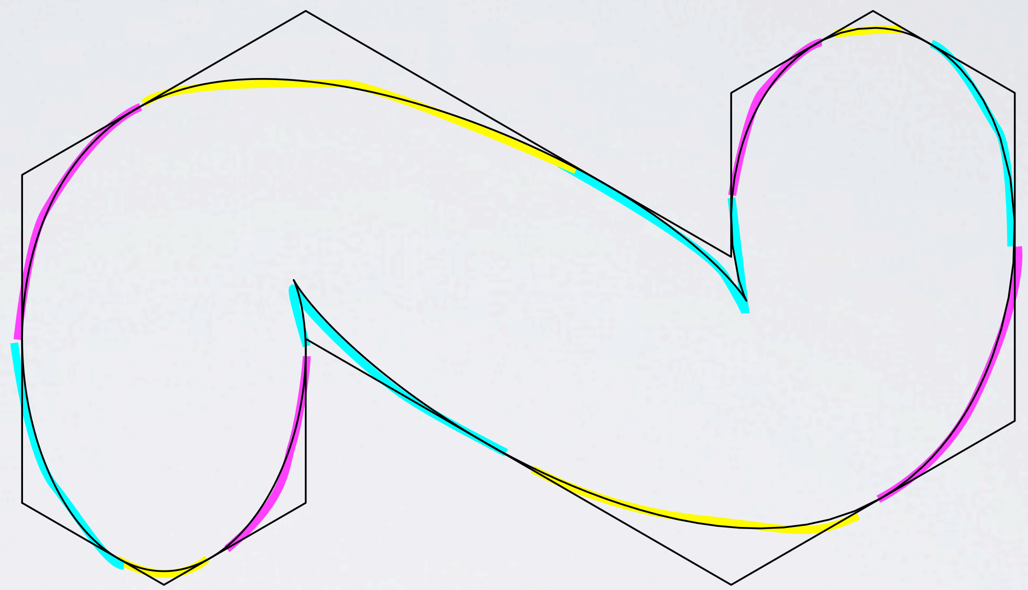
Kenyon-Okounkov

frozen boundary = free boundary

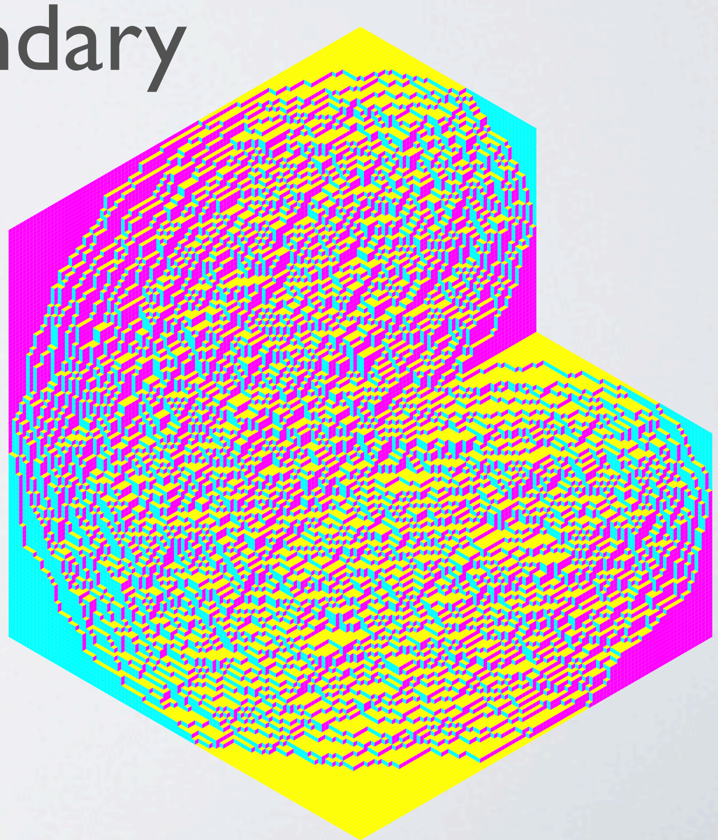
cloud curves

inscribed in polygons

(a solution to Euler-Lagrange exists)



polygonal boundary contour in coordinate directions → algebraic frozen boundary + frozen facets



deformation argument

through volume constraint

"It would be interesting to prove..."

ON THE VARIATIONAL PROBLEM

deSilva-Savin

Ω bdd Lipschitz domain $\qquad \nabla h \in N \qquad$ obstacles

admissible functions $\qquad h_{|\partial\Omega} = h_0 \qquad m := \inf h, \quad M := \sup h$

$$\min_h \int_\Omega \sigma(\nabla h)$$

C^1-regularity $\qquad h \in C^1$ away from the obstacles

(except on line segments in coordinate directions joining to $\partial\Omega$)

continuity $\qquad \Gamma(\nabla h)$ is continuous

$\Gamma : N \to \mathbb{S}^2$ one-point compactification of *int* N

$\partial N \leftrightarrow \{p\}$

THE BELTRAMI EQUATION

$$\bar{\partial} f = f \partial f$$

$f\colon U \to \mathbb{D}$

$U \subset \mathbb{C}$

bounded simply connected

(or finitely connected)

$f \in W^{1,2}_{loc}(U)$

→ $f \in C^{\infty}(U)$

(self-improves)

properness assumption: $f(z) \to \partial\mathbb{D}$, as $z \to \partial U$

$\nabla h \to \partial N$ at the boundary of U

(critical boundary values will force formation of frozen facets)

→ $f \in C(\bar{U})$

algebraic boundary

tangent relations

INVERSE MAP

$$f = B \circ g^{-1}$$

$g : \mathbb{D} \to U$ homeomorphism $\qquad \bar{\partial} g = -B(z)\overline{\partial g}$

$B : \mathbb{D} \to \mathbb{D}$ analytic + proper ➡ finite Blaschke product

$h = g + B\bar{g}$ is holomorphic in $\mathbb{D}$

$$(1 - |B|^2)g = h - B\bar{h}$$

$$h(\zeta) = B(\zeta)\overline{h(\zeta)} = \frac{\overline{h(\zeta)}}{B(\zeta)}$$

$h(z) - B(z)\overline{h(z)} \to 0, \quad z \to \zeta \in \partial\mathbb{D}$

extend by reflection $\quad h(z) = B(z)\overline{h(1/\bar{z})} \quad \longleftrightarrow \quad g(z) = g(1/\bar{z})$

$h(z) - h(1/\bar{z}) \to 0, \quad |z| \to 1 \qquad \bar{\partial} h = 0$ weakly across $\partial\mathbb{D}$

➡ h is a rational map of $\hat{\mathbb{C}}$

UNIVALENT POLYNOMIALS

$$B(z) = z^d, \quad d \geqslant 2$$

$$h(z) = z^d \overline{h(1/\bar{z})}, \quad h(0) = 0 \qquad h(z) = \alpha z + \ldots + \bar{\alpha} z^{d-1}$$

→ h is a *self-reflective* polynomial of degree *d-1*

$$g(z) = \frac{h(z) - z^d \overline{h(z)}}{1 - |z|^{2d}}$$

extends continously to the boundary

$$g(\zeta) = p(\zeta), \quad \zeta \in \partial\mathbb{D}$$

$$p(z) = h(z) - \frac{1}{d} z\, h'(z)$$

$p(\partial\mathbb{D})$ non self-crossing curve

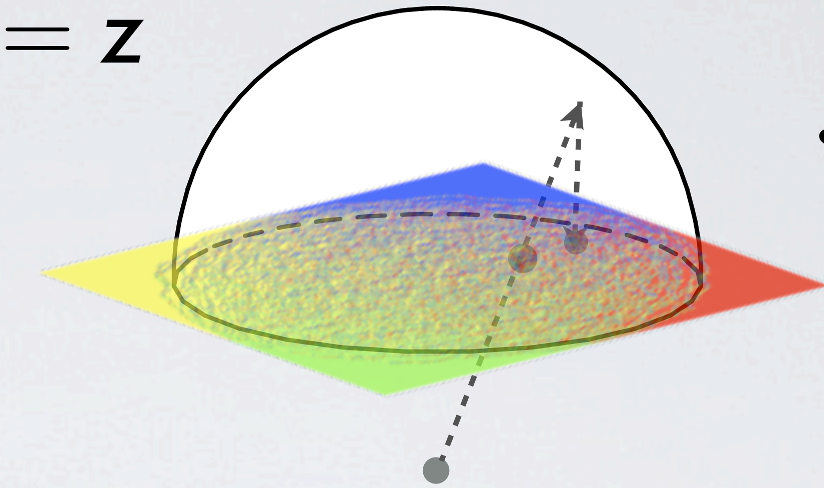
→ p is univalent in $\mathbb{D}$

$$p'(z) = (1 - \frac{1}{d})\alpha + \ldots + (1 - \frac{1}{d})\bar{\alpha} z^{d-2}$$

→ all *d*-2 critical points are on $\partial\mathbb{D}$ → $z^{d-2}\overline{p'(1/\bar{z})} = p'(z)$

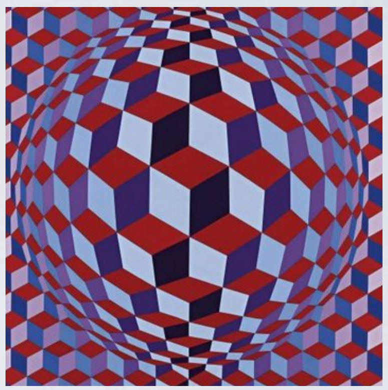
EXAMPLES

$$d = 2, \quad h(z) = 2z, \quad p(z) = z$$



$$g : \mathbb{D} \to \mathbb{D}$$

$$g(z) = \frac{2z}{1 + |z|^2}$$

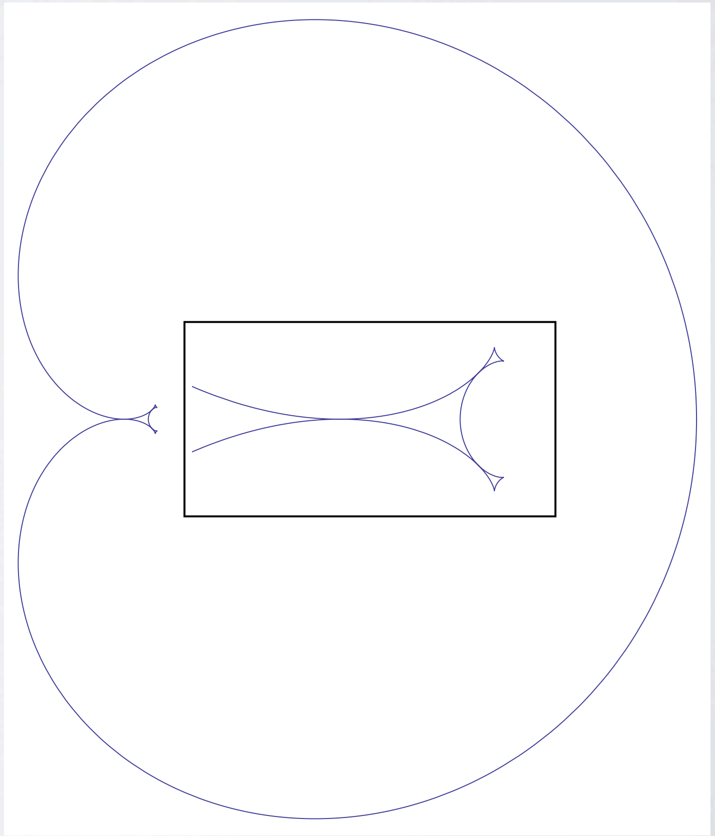


$$p'(z) = 1 - z^{d-2} \qquad h(z) = \frac{d}{d-1}(z - z^{d-1})$$

Suffridge curves: *d-3* double points

$$p(z) = \frac{z^5}{5} + \frac{4}{5}\sqrt{\frac{2}{3}}z^4 + \frac{6z^3}{5} + \frac{8}{5}\sqrt{2/3}z^2 + z$$

d=6



RATIONAL PARAMETRIZATION

$$g(z) = \frac{h(z) - B(z)\overline{h(z)}}{1 - |B(z)|^2} = \frac{\frac{h(z)}{B(z)} - \overline{h(z)}}{\frac{1}{B(z)} - \overline{B(z)}} = \frac{\frac{h(z)}{B(z)} - \frac{h(1/\bar{z})}{B(1/\bar{z})}}{\frac{1}{B(z)} - \frac{1}{B(1/\bar{z})}}$$

$\to r(\zeta), \quad z \to \zeta \in \partial\mathbb{D}$ $\qquad r = \frac{(h/B)'}{(1/B)'}$ $\qquad \overline{r(1/\bar{z})} = \frac{h'}{B'}$

$z\, r'(z)$ self-reflective with respect to *B(z)*

$\frac{B(z)}{z^2}\overline{r'(1/\bar{z})} = r'(z)$ tangent vector $i\zeta r'(\zeta) = i A(\zeta)\sqrt{B(\zeta)}$

real, sign changes at cusps

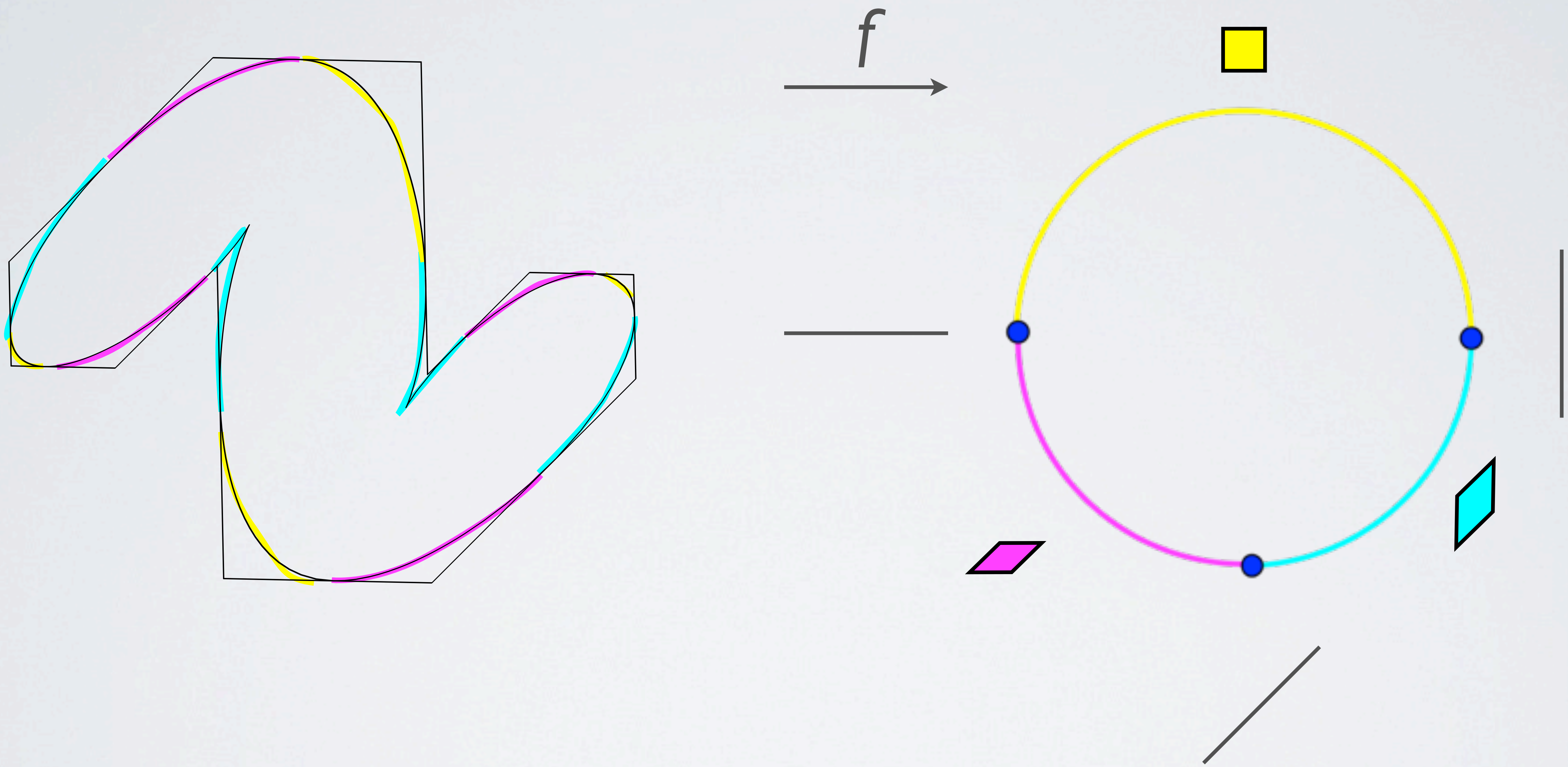
d/2 − 1/2 # cusps = 1

→ boundary is algebraic & locally convex except at *d*-2 cusps

→ $f = B \circ g^{-1}$ continuous up to the boundary

3d-3 real parameters

BOUNDARY RECONSTRUCTION



MULTIPLY CONNECTED CASE

$$f = B \circ g^{-1}$$

$\mathbb{D} \rightsquigarrow \mathcal{D}$ circle domain $\quad B\colon \mathcal{D} \to \mathbb{D}$ analytic and proper

B and h continuous up to the boundary as before

$$h(\zeta) = B(\zeta)\overline{h(\zeta)} = \frac{\overline{h(\zeta)}}{B(\zeta)}, \quad \zeta \in \partial\mathcal{D}$$

$(h, \overline{h/B})$

$(B, \overline{1/B})$ are meromorphic functions on the Schottky double $\hat{\mathcal{D}}$

j antiholomorphic involution

$$\tilde{r} = \left(\frac{h'}{B'}, \frac{\overline{(h/B)'}}{(1/B)'}\right) \qquad r = \overline{\tilde{r} \circ j}$$

meromorphic functions

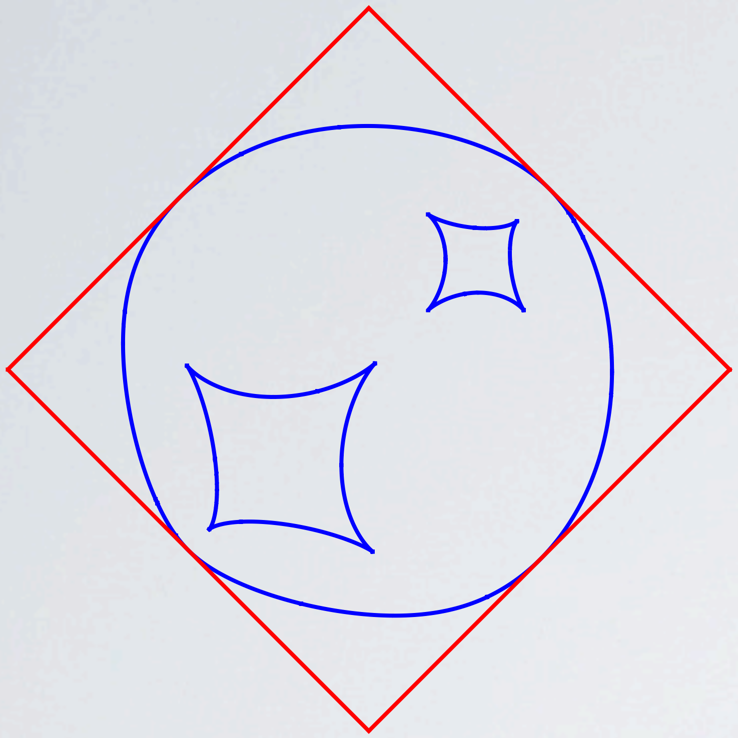
$$R(r, \tilde{r}) = 0$$

$$g(\zeta) = r(\zeta), \quad \zeta \in \partial\mathcal{D} \qquad R(g(\zeta), \overline{g(\zeta)}) = 0$$

algebraic boundary etc

Arctic curves of the octahedron equation

Philippe Di Francesco[1] and Rodrigo Soto-Garrido[2]



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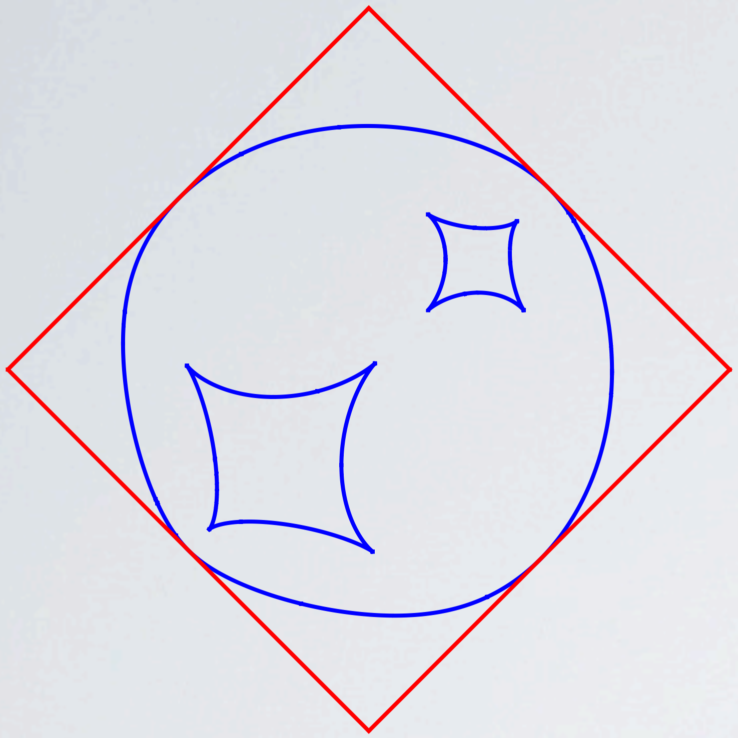

$$
\begin{aligned}
P(u,v) = {} & 603\,358\,073\,569\,688\,095\,393\,738\,000u^{14} + 1822\,971\,422\,522\,481\,873\,814\,304\,800vu^{13} + 302\,414\,835\,014\,281\,399\,576\,977\,600u^{13} \\
& + 7658\,013\,562\,515\,635\,323\,215\,886\,000v^2u^{12} + 626\,386\,479\,045\,976\,264\,625\,165\,760vu^{12} + 65\,648\,625\,922\,043\,130\,480\,407\,960u^{12} \\
& + 8502\,660\,801\,885\,990\,861\,442\,260\,800v^3u^{11} - 4016\,291\,377\,989\,674\,598\,197\,523\,840v^2u^{11} - 8955\,889\,812\,423\,159\,779\,663\,425\,824vu^{11} \\
& - 648\,516\,348\,371\,464\,166\,524\,636\,080u^{11} + 24\,870\,815\,123\,962\,290\,558\,144\,794\,000v^4u^{10} - 961\,218\,355\,287\,663\,519\,951\,292\,800v^3u^{10} \\
& - 6515\,407\,606\,857\,043\,381\,218\,037\,200v^2u^{10} - 172\,367\,781\,226\,698\,452\,854\,372\,560vu^{10} + 1099\,108\,080\,544\,208\,467\,044\,202\,281u^{10} \\
& + 8664\,424\,609\,796\,383\,599\,417\,068\,000v^5u^9 - 17\,028\,590\,399\,764\,390\,279\,389\,912\,000v^4u^9 - 11\,010\,604\,932\,056\,552\,215\,403\,730\,080v^3u^9 \\
& + 8631\,816\,024\,097\,405\,173\,283\,346\,160v^2u^9 + 6130\,342\,332\,781\,365\,103\,023\,636\,918vu^9 + 300\,038\,159\,586\,641\,951\,467\,587\,240u^9 \\
& + 48\,101\,067\,368\,389\,417\,583\,947\,124\,400v^6u^8 - 8874\,912\,221\,343\,735\,420\,641\,284\,800v^5u^8 - 30\,642\,500\,612\,723\,566\,034\,952\,420\,120v^4u^8 \\
& + 4089\,814\,979\,226\,593\,490\,453\,920\,400v^3u^8 + 2890\,833\,100\,949\,061\,663\,021\,542\,421v^2u^8 - 1537\,862\,122\,247\,709\,326\,673\,670\,200vu^8 \\
& - 1276\,791\,684\,735\,224\,437\,145\,235\,252u^8 + 165\,638\,160\,476\,224\,209\,249\,648\,000v^7u^7 - 15\,185\,547\,478\,970\,793\,846\,022\,129\,920v^6u^7 \\
& + 10\,619\,324\,480\,232\,243\,252\,222\,805\,440v^5u^7 + 17\,121\,927\,414\,923\,400\,963\,351\,428\,640v^4u^7 - 4642\,084\,019\,946\,561\,205\,079\,466\,936v^3u^7 \\
& - 4201\,308\,893\,745\,605\,384\,096\,673\,600v^2u^7 + 97\,128\,658\,780\,698\,750\,571\,038\,384vu^7 + 16\,658\,271\,644\,450\,437\,458\,125\,640u^7 \\
& + 48\,101\,067\,368\,389\,417\,583\,947\,124\,400v^8u^6 - 15\,185\,547\,478\,970\,793\,846\,022\,129\,920v^7u^6 - 54\,696\,534\,109\,775\,129\,942\,931\,200\,352v^6u^6 \\
& + 8498\,087\,480\,515\,562\,992\,290\,313\,440v^5u^6 + 20\,848\,735\,934\,263\,779\,279\,940\,738\,242v^4u^6 - 2928\,090\,072\,842\,649\,426\,783\,830\,400v^3u^6 \\
& - 1125\,942\,030\,946\,106\,640\,101\,862\,864v^2u^6 + 881\,693\,827\,811\,784\,667\,334\,364\,120vu^6 + 410\,818\,358\,444\,129\,895\,320\,450\,118u^6 \\
& + 8664\,424\,609\,796\,383\,599\,417\,068\,000v^9u^5 - 8874\,912\,221\,343\,735\,420\,641\,284\,800v^8u^5 + 10\,619\,324\,480\,232\,243\,252\,222\,805\,440v^7u^5 \\
& + 8498\,087\,480\,515\,562\,992\,290\,313\,440v^6u^5 - 16\,598\,910\,777\,434\,586\,615\,901\,305\,852v^5u^5 - 3118\,943\,690\,894\,703\,413\,913\,413\,040v^4u^5 \\
& + 4436\,727\,620\,735\,139\,576\,883\,870\,032v^3u^5 + 378\,779\,090\,210\,933\,672\,213\,353\,800v^2u^5 - 894\,275\,420\,028\,329\,313\,474\,734\,772vu^5 \\
& - 28\,143\,830\,188\,642\,461\,399\,955\,080u^5 + 24\,870\,815\,123\,962\,290\,558\,144\,794\,000v^{10}u^4 - 17\,028\,590\,764\,390\,279\,389\,912\,000v^9u^4 \\
& - 30\,642\,500\,723\,566\,034\,952\,420\,120v^8u^4 + 17\,121\,927\,923\,400\,963\,351\,428\,640v^7u^4 + 20\,848\,735\,263\,779\,279\,940\,738\,242v^6u^4 \\
& - 3118\,943\,690\,894\,703\,413\,913\,413\,040v^5u^4 - 6585\,025\,120\,215\,513\,060\,415\,620\,600v^4u^4 + 224\,576\,822\,600\,011\,254\,994\,156\,440v^3u^4 \\
& + 730\,062\,356\,407\,169\,871\,489\,508\,026v^2u^4 - 87\,999\,348\,446\,432\,687\,845\,418\,760vu^4 - 39\,991\,576\,579\,826\,072\,416\,315\,884u^4 \\
& + 8502\,660\,801\,885\,990\,861\,442\,260\,800v^{11}u^3 - 961\,218\,355\,287\,663\,519\,951\,292\,800v^{10}u^3 - 11\,010\,604\,932\,056\,552\,215\,403\,730\,080v^9u^3 \\
& + 4089\,814\,979\,226\,593\,490\,453\,920\,400v^8u^3 - 4642\,084\,019\,946\,561\,205\,079\,466\,936v^7u^3 - 2928\,090\,072\,842\,649\,426\,783\,830\,400v^6u^3 \\
& + 4436\,727\,620\,735\,139\,576\,883\,870\,032v^5u^3 + 224\,576\,822\,600\,011\,254\,994\,156\,440v^4u^3 - 771\,752\,886\,154\,129\,578\,670\,446\,744v^3u^3 \\
& + 54\,105\,975\,565\,681\,638\,845\,373\,840v^2u^3 + 158\,742\,939\,499\,283\,087\,522\,192\,736vu^3 + 3181\,828\,983\,737\,934\,822\,021\,000u^3 \\
& + 7658\,013\,562\,515\,635\,323\,215\,886\,000v^{12}u^2 - 4016\,291\,377\,989\,674\,598\,197\,523\,840v^{11}u^2 - 6515\,407\,606\,857\,043\,381\,218\,037\,200v^{10}u^2 \\
& + 8631\,816\,024\,097\,405\,173\,283\,346\,160v^9u^2 + 2890\,833\,100\,949\,061\,663\,021\,542\,421v^8u^2 - 4201\,308\,893\,745\,605\,384\,096\,673\,600v^7u^2 \\
& - 1125\,942\,030\,946\,106\,640\,101\,862\,864v^6u^2 + 378\,779\,090\,210\,933\,672\,213\,353\,800v^5u^2 + 730\,062\,356\,407\,169\,871\,489\,508\,026v^4u^2 \\
& + 54\,105\,975\,565\,681\,638\,845\,373\,840v^3u^2 - 205\,856\,416\,682\,486\,477\,443\,753\,704v^2u^2 - 4541\,013\,871\,098\,771\,634\,821\,000vu^2 \\
& - 417\,838\,190\,775\,940\,873\,949\,175u^2 + 1822\,971\,422\,522\,481\,873\,814\,304\,800v^{13}u + 626\,386\,479\,045\,976\,264\,625\,165\,760v^{12}u
\end{aligned}
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LIFE BEYOND THE ARCTIC CIRCLE