Some properties of Marcinkiewicz means with respect to Walsh system[1]

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Introduction

Let denote by

$$\mathbb{Z}_2$$

the discrete cyclic group of order 2, that is $\mathbb{Z}_2 = \{0, 1\}$, the group operation is the modulo 2 addition, every subset is open. Haar measure on $\mathbb{Z}_2$ is given in the way that the measure of a singleton is $1/2$.

The Walsh group:

$$G := \mathop{\times}_{k=0}^{\infty} \mathbb{Z}_2.$$

The elements of G are of the form

$$x = (x_0, x_1, ..., x_k, ...)$$

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The group operation on G is the coordinate-wise addition, the measure (denoted by μ) and the topology are the product measure and topology.

Fine's map:

For $x \in G$ we define $|x|$ by $|x| := \sum_{j=0}^{\infty} x_j 2^{-j-1}$.

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Rademacher functions:

$$r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbb{N}).$$

If $n \in \mathbb{N}$, then

$$n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0,1\} \ (i \in \mathbb{N}).$$

Let be the order of n

$$|n| := \max\{j \in \mathbb{N} : n_j \neq 0\}.$$

Walsh-Paley functions:

$$w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k}$$

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Walsh-Kaczmarz functions:

$$\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{|n|-1} (r_{|n|-1-k}(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}},$$

The Walsh-Kaczmarz system:

$$\kappa := (\kappa_n : n \in \mathbb{N}).$$

It is well known that

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{w_n : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbb{N}$ and $\kappa_0 = w_0$.

Walsh-Kaczmarz functions:

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A relation between Walsh-Kaczmarz functions and Walsh-Paley functions :

The transformation $\tau_A \colon G \to G$ $(A \in \mathbb{N})$. given by V. A. Skvortsov

$$\tau_A(x) := (x_{A-1}, x_{A-2}, ..., x_1, x_0, x_A, x_{A+1}, ...)$$

The relation

$$\kappa_n(x) = r_{|n|}(x) w_{n-2^{|n|}}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in G).$$

Fourier coefficients, partial sums, Dirichlet kernels, Fejér means, Fejér kernels:

$$\hat{f}^{\psi}(n) := \int_G f\psi_n, \quad S_n^{\psi} f := \sum_{k=0}^{n-1} \hat{f}^{\psi}(k)\psi_k,$$

$$D_n^{\psi} := \sum_{k=0}^{n-1} \psi_k, \quad \sigma_n^{\psi} f := \frac{1}{n}\sum_{k=0}^{n-1} S_k^{\psi} f,$$

$$K_n^{\psi} := \frac{1}{n}\sum_{k=0}^{n-1} D_k^{\psi},$$

where $\psi = w$ or κ.

Let L_p denote the usual Lebesgue space with the norm (or quasinorm) $\|.\|_p \quad (0 < p < \infty)$.
The space weak-L_p consists of all measurable function f for which

$$\|f\|_{weak-L_p} := \sup_{\lambda>0} \lambda\mu(|f| > \lambda)^{1/p} < \infty.$$

Let the operator $T\colon H_p \to L_p$. The operator T is of type (H_p, L_p) if there exists a constant $c_p > 0$ such that

$$\|Tf\|_p \leq c_p\|f\|_{H_p} \quad \text{for all } f \in H_p.$$

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Let the operator $T\colon H_p \to \text{weak} - L_p$. The operator T is of type weak-(H_p, L_p) if there exists a constant $c_p > 0$ such that

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Definitions and notations

Two-dimensional systems: The Kronecker product $(\psi_{n,m} : n, m \in \mathbf{N})$ of two Walsh (-Kaczmarz) system, where

$$\psi_{n,m}\left(x^1, x^2\right) = \psi_n\left(x^1\right)\psi_m\left(x^2\right),$$

where $\psi = w$ or κ.

Two-dimensional Walsh-(Kaczmarz-)Fourier coefficient:

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Rectangular partial sum of the Walsh-Fourier series, the Marcinkiewicz-Fejér means

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Some historical notes

- I. Marcinkiewicz (1939) for $f \in L \log L([0, 2\pi]^2)$ and for trigonometric system the mean

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converge a.e. to f as $n \to \infty$ Ann. Scuola Norm. Sup. Pisa 8 (1939) 149-160.

- L.V. Zhizhiashvili (1968) improved this result for $f \in L_1([0, 2\pi]^2)$ the (C, α)-mean of the cubical partial sums converge a.e. to f as $n \to \infty$ Izv. Akad. Nauk USSR Ser Math. 32 (1968) 1112-1122.

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Historical notes on the Walsh-(Kaczmarz-)Marcinkiewicz means

- F. Weisz (2001)The a. e. convergence of Walsh-Marcinkiewicz means of integrable functions. Appr. Theory Appl. 17 (2001) 32-44.
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- K. Nagy (2006) The a. e. convergence of Walsh-Kaczmarz-Marcinkiewicz means of integrable functions. J. Approx. Theory 142 (2006) 138-165.

For f we consider the maximal operator

$$\mathcal{M}^{\psi,*}f(x^1,x^2)=\sup_{n\in\mathbb{P}}|\mathcal{M}_n^{\psi}(f;x^1,x^2)|.$$

Connecting results: the maximal operator $\mathcal{M}^*$ is of weak type $(1,1)$ and of type (p,p) for all $1<p\leq\infty$.

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- G. Gát, U. Goginava and K. Nagy (2009) The maximal operator $\mathcal{M}^{\kappa,*}$ is bounded from martingale Hardy space H_p to the space L_p for $1 \geq p > 2/3$. Studia Sci. Math. Hung. 46 (2009) 399-421
- U. Goginava (2006): in the endpoint case $p = 2/3$ the boundedness does not hold. Walsh-Paley system. East J. Approx. 12(3) (2006), 295-302.
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Historical notes on the Walsh-(Kaczmarz-)Marcinkiewicz means

What does happen in the end point case $p = 2/3$?

Direction 1:

- U. Goginava (2008): for Walsh-Paley system There exists a martingale $f \in H_{2/3}$ such that

$$\|\mathcal{M}^{w,*} f\|_{2/3} = +\infty.$$

Acta Math. Sinica (2008)

- U. Goginava and K. Nagy (2009): for Walsh-Kaczmarz system analogical result Publ. Math. Debrecen 75 (1-2) (2009) 95-104.

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Direction 2:

Define the maximal operator $\tilde{\mathcal{M}}^*$ by

$$\tilde{\mathcal{M}}^* f := \sup_{n\in\mathbb{P}} \frac{|\mathcal{M}_n f|}{\log^{3/2}(n+1)}.$$

- K. Nagy (2011) The maximal operator $\tilde{\mathcal{M}}^{w,*}$ is bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$.
Moreover, the following holds:
Let $\varphi : \mathbb{P} \to [1,\infty)$ be a non-decreasing function satisfying the condition

$$\overline{\lim_{n\to\infty}} \frac{\log^{3/2}(n+1)}{\varphi(n)} = +\infty$$

Then the maximal operator $\sup_{n\in\mathbb{P}} \frac{|\mathcal{M}_n^{\kappa} f|}{\varphi(n)}$ is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$.

Publ. Math. Debrecen 78(3-4) (2011) 633-646.

Direction 2.

The order of the deviant behaviour of the nth Walsh-Marcinkiewicz means is

$$\log^{3/2}(n+1).$$

-K. Nagy (2015): for Walsh-Kaczmarz system analogical result

Mathematical Inequalities and Applications 18 (1) (2015) 97-110.

Direction 3:

- U. Goginava (2008): The maximal operator $\mathcal{M}^{w,*}$ is bounded from the Hardy space $H_{2/3}$ to the space weak-$L_{2/3}$. J. Approx. Theory 154 (2008) 161-180.

Theorem (U. Goginava, K. Nagy (2016))

The maximal operator $\mathcal{M}^{\kappa,*}$ is bounded from the Hardy space $H_{2/3}$ to the space weak-$L_{2/3}$.

Acta Mathematica Scientia 36 (2) (2016) 359-370.

Direction 4:

- K. Nagy, G. Tephnadze (2014): a necessary and sufficient condition for the convergence of Walsh-Marcinkiewicz means in terms of the modulus of continuity on the Hardy space $H_{2/3}\left(G^2\right)$.

Kyoto Journal of Mathematics 54 (3) (2014) 641-652.

- K. Nagy, G. Tephnadze (2014): analogical results for Walsh-Kaczmarz system.

Bulletin of TICMI 18 (1) (2014) 110-121.

Let us define the modulus of continuity in the Hardy space H_p by

$$\omega\left(\frac{1}{2^n}, f\right)_{H_p} := \|f - S_{2^n,2^n}(f)\|_{H_p}$$

Theorem

a) Let

$$\omega\left(\frac{1}{2^k}, f\right)_{H_{2/3}} = o\left(\frac{1}{k^{3/2}}\right),$$

as $k \to \infty$. *Then*

$$\|\mathcal{M}_n(f) - f\|_{H_{2/3}} \to 0, \text{ when } n \to \infty.$$

b) There exists a martingale $f \in H_{2/3}$, *for which*

$$\omega\left(\frac{1}{2^{2^k}}, f\right)_{H_{2/3}} = O\left(\frac{1}{2^{3k/2}}\right),$$

as $k \to \infty$ *and*

$$\|\mathcal{M}_n(f) - f\|_{2/3} \not\to 0 \text{ as } n \to \infty.$$

Direction 5:

- K. Nagy, G. Tephnadze (2016): Strong convergence theorem for Walsh system.

Theorem (K. Nagy, G. Tephnadze (2016))

There exists an absolute constant c, *such that*

$$\frac{1}{\log n}\sum_{m=1}^{n}\frac{\left\|\mathcal{M}_m^w(f)\right\|_{H_{2/3}}^{2/3}}{m}\leq c\left\|f\right\|_{H_{2/3}}^{2/3},$$

for all $f\in H_{2/3}\left(G^2\right)$.

Mathematical Inequalities and Applications 19 (1) (2016) 185-195.

for Walsh-Kaczmarz system it is open problem.

New results in the case $0 < p < 2/3$

What does happen in the case $0 < p < 2/3$?

Define the maximal operator $\widetilde{\sigma}^{*,p}$ by

$$\widetilde{\mathcal{M}}^{*,p}(f) := \sup_{n\geq 1}\left|\frac{\mathcal{M}_n(f)}{n^{2/p-3}}\right|,$$

Theorem (K. Nagy, G. Tephnadze)

a) Let $0 < p < 2/3$. Then the maximal operator $\widetilde{\mathcal{M}}^{*,p}$ is bounded from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$.

b) Let $\varphi\colon \mathbb{N} \to [1,\infty)$ be a non-decreasing function, satisfying the condition

$$\lim_{n\to\infty}\frac{n^{2/p-3}}{\varphi(n)} = +\infty. \qquad (1)$$

Then

$$\sup_{n\in\mathbb{N}}\left\|\frac{\mathcal{M}_n f}{\varphi(n)}\right\|_{weak-L_p} = \infty.$$

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Then

$$\sup_{n \in \mathbb{N}} \left\| \frac{\mathcal{M}_n f}{\varphi(n)} \right\|_{weak-L_p} = \infty.$$

New results in the case $0 < p < 2/3$

That is, the exact order of deviant behaviour of the n-th Walsh-Marcikiewicz mean is calculated in Hardy space H_p for $0 < p < 2/3$. It is

$$n^{2/p-3}.$$

- K. Nagy, G. Tephnadze (2014): for Walsh-Paley system. Central European Journal of Mathematics 12 (8) (2014) 1214-1228.

- K. Nagy, G. Tephnadze (2016): for Walsh-Kaczmarz system. Acta Mathematica Hungarica 149 (2) (2016) 346-374.

New results in the case $0 < p < 2/3$

After this two applications were given.

Application 1: A necessary and sufficient condition for the convergence of Walsh-Marcinkiewicz means in terms of the modulus of continuity on the Hardy space $H_p\left(G^2\right)$ for $0 < p < 2/3$.

Application 2: A strong convergence theorem.

Theorem (K. Nagy, G. Tephnadze)

a) Let $1/2 < p < 2/3$, $f \in H_p\left(G^2\right)$ *and*

$$\omega\left(\frac{1}{2^k}, f\right)_{H_p} = o\left(\frac{1}{2^{k(2/p-3)}}\right),$$

as $k \to \infty$*. Then*

$$\left\|\mathcal{M}_n\left(f\right) - f\right\|_{H_p} \to 0, \text{ when } n \to \infty.$$

b) Let $0 < p < 2/3$*. Then there exists a martingale* $f \in H_p(G^2)$*, such that*

$$\omega\left(\frac{1}{2^k}, f\right)_{H_p} = O\left(\frac{1}{2^{k(2/p-3)}}\right),$$

as $k \to \infty$ *and*

$$\left\|\mathcal{M}_n\left(f\right) - f\right\|_{weak-L_p} \not\to 0 \text{ as } n \to \infty.$$

New results in the case $0 < p < 2/3$

Theorem (K. Nagy, G. Tephnadze)

a) Let $0 < p < 2/3$. Then there exists an absolute constant c_p, such that

$$\sum_{m=1}^{\infty} \frac{\|\mathcal{M}_m f\|_{H_p}^p}{m^{3-3p}} \leq c_p \|f\|_{H_p}^p$$

for all $f \in H_p\left(G^2\right)$.

b) Let $0 < p < 2/3$ and $\Phi : \mathbb{N}_+ \to [1, \infty)$ be any non-decreasing function, satisfying the conditions $\Phi(n) \uparrow \infty$ and

$$\overline{\lim_{k\to\infty}} \frac{2^{k(3-3p)}}{\Phi\left(2^k\right)} = \infty.$$

Then there exists a martingale $f \in H_p\left(G^2\right)$, such that

$$\sum_{m=1}^{\infty} \frac{\|\mathcal{M}_m f\|_{weak-L_p}^p}{\Phi(m)} = \infty.$$

Thanks for your attention!