On approximation by multivariate Kantorovich–Kotelnikov sampling operators

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Basic notation

- L_p denotes $L_p(\mathbb{R}^d)$, $1 \le p \le \infty$, with the usual norm $\|f\|_p = \|f\|_{L_p(\mathbb{R}^d)} < \infty$
- If $f \in L_1$, then its Fourier transform is

$$\widehat{f}(\xi) = \int_{\mathbb{R}^d} f(x) e^{-2\pi i (x,\xi)}\, dx$$

- The modulus of smoothness $\omega_r(f,\delta)_p$ of order $r \in \mathbb{N}$ for a function $f \in L_p$ is defined by

$$\omega_r(f,\delta)_p = \sup_{|h|\le\delta,\, h\in\mathbb{R}^d} \|\Delta_h^r f\|_p$$

where

$$\Delta_h^r f(x) = \Delta_h^1 \Delta_h^{r-1} f(x), \quad \Delta_h^1 f(x) = f(x+h) - f(x)$$

- The Kantorovich–Kotelnikov operator is an operator of the form

$$K_w(f,\varphi;x)=\sum_{k\in\mathbb{Z}}\left(w\int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)du\right)\varphi(wx-k),\quad x\in\mathbb{R},\quad w>0$$

where $f:\mathbb{R}\to\mathbb{C}$ is a locally integrable function and φ is an appropriate kernel

- The operator K_w has several advantages over the generalized sampling operators

$$S_w(f,\varphi;x)=\sum_{k\in\mathbb{Z}} f\left(\frac{k}{w}\right)\varphi(wx-k),\quad x\in\mathbb{R},\quad w>0$$

  - Approximation and reconstruction not necessary continuous signals
  - Reducing the so-called time-jitter errors
  - Better approximation order

History

- We consider the *generalized Kantorovich–Kotelnikov sampling operator*

$$Q_{M^j}(f,\varphi,\widetilde{\varphi};x) = \sum_{k\in\mathbb{Z}^d} \left(m^j \int_{\mathbb{R}^d} f(u)\widetilde{\varphi}(M^j u + k)du \right) \varphi(M^j x + k), \quad j \in \mathbb{Z}$$

where M is a dilation matrix, $m = |\det M|$, and $\widetilde{\varphi}$ and φ are appropriate functions.

  - If $d = 1$ and $\widetilde{\varphi}(x) = \chi_{[0,1]}(x)$, then Q_{M^j} represents the standard operator K_{m^j}.

R.Q. Jia (1995, 2003, 2010)

P.L. Butzer, R.Q. Jia (2000)

J.J. Lei, R.Q. Jia, E.W. Cheney (1997)

C. Bardaro, P.L. Butzer, R.L. Stens, G. Vinti (2007)

G. Vinti, L. Zampogni (2009, 2014)

A. Krivoshein, M. Skopina (2011, 2016)

M. Skopina (2014)

D. Costarelli, G. Vinti (2014, 2015)

F. Cluni, D. Costarelli, A.M. Minotti, G. Vinti (2015)

O. Orlova, G. Tamberg (2016)

- **Theorem** (R.Q. Jia (2003)) *If* φ *and* $\widetilde{\varphi}$ *are compactly supported,* $\varphi \in L_p(\mathbb{R}^d)$, $\widetilde{\varphi} \in L_q(\mathbb{R}^d)$, $1/p + 1/q = 1$, *M is an isotropic dilation matrix, and* Q_I *reproduces polynomials of degree* $n-1$, *i.e.* $Q_I P = P$ *for all* $P \in \Pi_{n-1}$, *then for any* $f \in L_p(\mathbb{R})$, $1 \le p \le \infty$, *and* $j \in \mathbb{N}$, *we have*

$$\|f - Q_{M^j}(f, \varphi, \widetilde{\varphi})\|_{L_p(\mathbb{R}^d)} \le C\omega_n(f, m^{-\frac{j}{d}})_p$$

- **Question:** What do we have for band-limited functions φ, e.g., $\mathrm{sinc}(x) = \frac{\sin \pi x}{\pi x}$?

  Recall that the classical Whittaker-Shannon-Kotelnikov sampling series is given by

$$S_w(f, ; x) = \sum_{k \in \mathbb{Z}} f\left(\frac{k}{w}\right) \mathrm{sinc}(wx - k)$$

  Then the Kanorovich-type version of S_w is defined by

$$K_w(f; x) = \sum_{k \in \mathbb{Z}} \left(w \int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)du \right) \mathrm{sinc}(wx - k)$$

Known results

- As a rule, to study the operator

$$K_w(f,\varphi;x) = \sum_{k\in\mathbb{Z}} \left(w \int_{\frac{k}{w}}^{\frac{k+1}{w}} f(u)du \right) \varphi(wx-k)$$

  one supposes that

  - $\varphi \in L_1(\mathbb{R})$
  - for every $u \in \mathbb{R}$,

$$\sum_{k\in\mathbb{Z}} \varphi(u-k) = 1$$

  - for some $\beta \geq 0$,

$$\sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{R}} |\varphi(u-k)||u-k|^\beta < \infty$$

- C. Bardaro, P.L. Butzer, R.L. Stens, G. Vinti (2007):
  Theorem. For every $f \in L_p(\mathbb{R})$, $1 \leq p < \infty$, we have

$$\lim_{w\to\infty} \|f - K_w(f,\varphi)\|_{L_p(\mathbb{R})} = 0$$

- D. Costarelli, G. Vinti (2014):
    - $\varphi \in L_1(\mathbb{R})$ and is bounded in a neighborhood of 0
    - For some $\mu > 0$,

$$\sum_{k\in\mathbb{Z}} \varphi(wx-k) = 1 + \mathcal{O}(w^{-\mu}) \quad \text{as} \quad w \to +\infty$$

    - For some $\beta > 0$,

$$\sup_{u\in\mathbb{R}} \sum_{k\in\mathbb{R}} |\varphi(u-k)||u-k|^{\beta} < \infty$$

    - There exists $\alpha > 0$ such that, for every $N > 0$,

$$\int_{|u|>N} w|\varphi(wu)|du = \mathcal{O}(w^{-\alpha}) \quad \text{as} \quad w \to +\infty$$

    - For some $0 < \nu \le 1$,

$$\int_{\mathbb{R}} |\varphi(u)||u|^{\nu} du < \infty$$

- **Theorem.** *For any $f \in L_p(\mathbb{R}) \cap \mathrm{Lip}(\nu)$, $1 \le p \le \infty$, $0 < \nu \le 1$, we have*

$$\|f - K_w(f,\varphi)\|_{L_p(\mathbb{R})} = \mathcal{O}(w^{-\epsilon}) \quad \textit{as} \quad w \to +\infty$$

where $\epsilon = \min\{\nu, \mu, \alpha\}$.

- O. Orlova, G. Tamberg (2016):

$$Q_w(f,\varphi,\widetilde{\varphi};x)=\sum_{k\in\mathbb{Z}}\left(w\int_{\mathbb{R}} f(u)\widetilde{\varphi}(k-wu)du\right)\varphi(wx-k)$$

where

  - $\varphi,\widetilde{\varphi}\in L_1(\mathbb{R})$
  - $\sup\limits_{u\in\mathbb{R}}\sum\limits_{k\in\mathbb{R}}|\varphi(u-k)|<\infty$
  - $\sum\limits_{k\in\mathbb{Z}}\varphi(u-k)=1,\ u\in\mathbb{R},$ and $\int_{\mathbb{R}}\widetilde{\varphi}(x)dx=1$

- **Theorem.** *Let $\lambda,\widetilde{\lambda}\in C(\mathbb{R})$, $\lambda(0)=\widetilde{\lambda}(0)=1$, and $\widetilde{\lambda}(2k)=0$, $k\in\mathbb{Z}$. Suppose*

$$\varphi(x)=\int_0^1\lambda(u)\cos(\pi xu)du,\qquad \widetilde{\varphi}(x)=\int_0^\infty\widetilde{\lambda}(u)\cos(\pi xu)du$$

and for some $r\in\mathbb{N}$

$$\lambda(u)\widetilde{\lambda}(u)=1-\sum_{j=r}^\infty c_j u^{2j},\quad \sum_{j=r}^\infty|c_j|<\infty$$

Then for any $f\in L_p(\mathbb{R})$, $1\le p\le\infty$, and $w>0$,

$$\|f-Q_w(f,\varphi,\widetilde{\varphi})\|_{L_p(\mathbb{R})}\le C\omega_{2r}(f,1/w)_p$$

- We consider the generalized Kantorovich – Kotelnikov sampling operator

$$Q_{M^j}(f,\varphi,\widetilde{\varphi};x)=\sum_{k\in\mathbb{Z}^d}\left(m^j\int_{\mathbb{R}^d}f(u)\widetilde{\varphi}(M^ju+k)du\right)\varphi(M^jx+k),\quad j\in\mathbb{Z}$$

- Denote by $\mathcal{B}=\mathcal{B}(\mathbb{R}^d)$ the class of functions φ given by

$$\varphi(x)=\int\limits_{\mathbb{R}^d}\theta(\xi)e^{2\pi i(x,\xi)}\,d\xi$$

where θ is supported in a parallelepiped $\Pi:=[a_1,b_1]\times\cdots\times[a_d,b_d]$ and such that $\theta\big|_\Pi\in C^d(\Pi)$.

- Denote by $\mathcal{L}_p$, $1 \le p \le \infty$, the set

$$\mathcal{L}_p := \left\{ \varphi \in L_p \,:\, \|\varphi\|_{\mathcal{L}_p} := \left\| \sum_{k \in \mathbb{Z}^d} |\varphi(\cdot + k)| \right\|_{L_p(\mathbb{T}^d)} < \infty \right\}$$

The simple properties are:

  - If $1 \le q \le p \le \infty$, then

$$\mathcal{L}_1 = L_1, \quad \mathcal{L}_p \subset L_p, \quad \mathcal{L}_p \subset \mathcal{L}_q$$

  - If $\varphi \in L_p$ and compactly supported, then $\varphi \in \mathcal{L}_p$ for $p \ge 1$.
  - If φ decays fast enough, i.e. there exist constants $C > 0$ and $\varepsilon > 0$ such that

$$|\varphi(x)| \le \frac{C}{(1 + |x|)^{d+\varepsilon}} \quad \text{for all} \quad x \in \mathbb{R}^d$$

then $\varphi \in \mathcal{L}_\infty$.

- Recall that a real $d \times d$ matrix M is called a *dilation matrix* if all eigenvalues of M are bigger than 1 in modulus.

  Recall also that

  $$\|M^{-j}\| \le C_{M,\vartheta}\, \vartheta^{-j}, \quad j \in \mathbb{Z}_+$$

  for every positive number ϑ which is smaller in modulus than any eigenvalue of M. In particular, we can take $\vartheta > 1$, then

  $$\lim_{j \to +\infty} \|M^{-j}\| = 0$$

Def. The functions $\widetilde{\varphi}$ and φ are said to be *strictly compatible* if there exists $\delta \in (0, 1/2)$ such that

$$\overline{\widehat{\varphi}}(\xi)\widehat{\widetilde{\varphi}}(\xi) = 1 \quad \text{a.e. on} \quad \{|\xi| < \delta\}$$

and

$$\widehat{\varphi}(\xi) = 0 \quad \text{a.e. on} \quad \{|l - \xi| < \delta\} \quad \text{for all} \quad l \in \mathbb{Z}^d \setminus \{0\}$$

Theorem 1. *Let $f \in L_p$, $1 \le p \le \infty$, and $n \in \mathbb{N}$. Suppose that φ and $\widetilde{\varphi}$ are strictly compatible and*

(i) $\varphi \in \mathcal{B}$ *and* $\widetilde{\varphi} \in \mathcal{B} \cup \mathcal{L}_{\frac{p}{p-1}}$ *in the case* $1 < p < \infty$,

(ii) $\varphi \in \mathcal{B} \cap L_1$ *and* $\widetilde{\varphi} \in \mathcal{L}_\infty$ *in the case* $p = 1$,

(iii) $\varphi \in \mathcal{L}_\infty$ *and* $\widetilde{\varphi} \in L_1$ *in the case* $p = \infty$.

Then

$$\|f - Q_{M^j}(f, \varphi, \widetilde{\varphi})\|_p \le C\omega_n \left(f, \|M^{-j}\|\right)_p$$

where C does not depend on f and j.

Main results

Theorem 2. *Let* $f \in L_p$, $1 \le p \le \infty$, *and* $n \in \mathbb{N}$. *Suppose* $\widehat{\varphi}, \widehat{\widetilde{\varphi}} \in C^{n+d+1}(B_\delta)$ *for some* $\delta > 0$, $D^\beta(1 - \widehat{\varphi}\overline{\widehat{\widetilde{\varphi}}})(\mathbf{0}) = 0$ *for all* $\beta \in \mathbb{Z}_+^d$, $[\beta] < n$, $\operatorname{supp} \widehat{\varphi} \subset B_{1-\varepsilon}$ *for some* $\varepsilon \in (0,1)$, *and*

(i) $\varphi \in \mathcal{B}$ *and* $\widetilde{\varphi} \in \mathcal{B} \cup \mathcal{L}_{\frac{p}{p-1}}$ *in the case* $1 < p < \infty$,

(ii) $\varphi \in \mathcal{B} \cap L_1$ *and* $\widetilde{\varphi} \in \mathcal{L}_\infty$ *in the case* $p = 1$,

(iii) $\varphi \in \mathcal{B} \cap \mathcal{L}_\infty$ *and* $\widetilde{\varphi} \in L_1$ *in the case* $p = \infty$.

Then

$$\|f - Q_{M^j}(f, \varphi, \widetilde{\varphi})\|_p \le C\omega_n\left(f, \|M^{-j}\|\right)_p$$

where C *does not depend on* f *and* j.

Let

$$\varphi(x) = \mathrm{sinc}(x) := \prod_{\nu=1}^{d} \frac{\sin(\pi x_\nu)}{\pi x_\nu} \quad \text{and} \quad \widetilde{\varphi}(x) = \frac{1}{\mathrm{mes}\, U}\chi_U(x)$$

Proposition 1. *Let $f \in L_p$, $1 < p < \infty$, and let U be a bounded measured subset of $\mathbb{R}^d$. Then*

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \frac{m^j}{\mathrm{mes}\, U} \int_{M^{-j}U} f(-M^{-j}k + t)\, dt\ \mathrm{sinc}(M^j \cdot + k) \right\|_p \le C\omega_1(f, \|M^{-j}\|)_p$$

where C does not depend on f and j.

If, in addition, U is symmetric with respect to the origin, then the modulus of continuity $\omega_1(f, \|M^{-j}\|)_p$ can be replaced by $\omega_2(f, \|M^{-j}\|)_p$.

Remark 1. Proposition 1 is valid for all $f \in L_p$, $1 \le p \le \infty$, if we replace $\mathrm{sinc}(x)$ by $\mathrm{sinc}^2(x)$. The same conclusion holds for all propositions presented below.

Remark 2. Note that Proposition 1 gives an answer to the question posed by C. Bardaro, P.L. Butzer, R.L. Stens, G. Vinti (2007)

Special cases

Proposition 2. *Let* $f \in L_p$, $1 < p < \infty$, $n \in \mathbb{N}$, *and let* $U \subset \mathbb{R}^d$. *Then there exists a finite set of numbers* $\{a_l\}_{l \in \mathbb{Z}^d} \subset \mathbb{C}$ *depending only on* d, n, *and* U *such that for*

$$\varphi(x) = \sum_l a_l \operatorname{sinc}(x + l) \tag{1}$$

we have

$$\left\| f - \sum_{k \in \mathbb{Z}^d} \frac{m^j}{\operatorname{mes} U} \int\limits_{M^{-j}U} f(-M^{-j}k + t)\, dt\, \varphi(M^j \cdot + k) \right\|_p \le C\omega_n(f, \|M^{-j}\|)_p$$

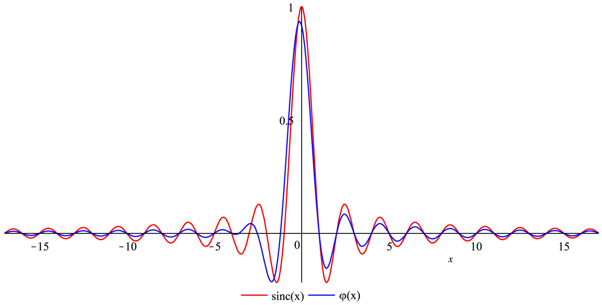
where C *does not depend on* f *and* j.

Example 1. Let $d = 1$, $U = [-1/2, 1/2]$, $M = 2$, and

$$\varphi(x) = \frac{11}{12}\,\mathrm{sinc}(x) + \frac{5}{24}\,\mathrm{sinc}(x+1) - \frac{1}{6}\,\mathrm{sinc}(x+2) + \frac{1}{24}\,\mathrm{sinc}(x+3)$$

Then

$$\left\| f - \sum_{k\in\mathbb{Z}} 2^j \int_{-2^{-j-1}}^{2^{-j-1}} f(-2^{-j}k + t)\, dt\, \varphi(2^j \cdot + k) \right\|_p \le C\omega_4(f, 2^{-j})_p$$



Example 2. Let $d = 2$, $U = B_1$, and

$$\varphi(x_1, x_2) = \frac{3}{2}\operatorname{sinc} x_1 \operatorname{sinc} x_2 - \frac{\operatorname{sinc} x_2}{8}(5\operatorname{sinc}(x_1+1) - 4\operatorname{sinc}(x_1+2) + \operatorname{sinc}(x_1+3)) - \frac{\operatorname{sinc} x_1}{8}(5\operatorname{sinc}(x_2+1) - 4\operatorname{sinc}(x_2+2) + \operatorname{sinc}(x_2+3))$$

Then

$$\left\| f - \sum_{k\in\mathbb{Z}^2} \frac{m^j}{\pi} \int\limits_{M^{-j}B_1} f(-M^{-j}k + t)\, dt\, \varphi(M^j \cdot + k) \right\|_p \le C\omega_4(f, \|M^{-j}\|)_p$$

The following estimate is a trivial consequence of Proposition 2:

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \sum_{l} \frac{a_l m^j}{\text{mes}\, U} \int\limits_{M^{-j}(U+l)} f(-M^{-j}k+t)\,dt\ \text{sinc}(M^j\cdot+k) \right\|_p \le C\omega_n(f, \|M^{-j}\|)_p$$

where $\{a_l\}$ are the same as in (1).

For functions $\varphi(x) \ne \text{sinc}(x)$, we have the following result:

Proposition 3. *Let $f \in L_p$, $1 < p < \infty$, $n \in \mathbb{N}$, and let $U \subset \mathbb{R}^d$. Suppose that $\varphi \in \mathcal{B}$, $\widehat{\varphi} \in C^{n+d+1}(B_\delta)$ for some $\delta > 0$, and $\operatorname{supp}\widehat{\varphi} \subset B_{1-\varepsilon}$ for some $\varepsilon \in (0,1)$. Then there exists a finite set of numbers $\{b_l\}_{l\in\mathbb{Z}^d} \subset \mathbb{C}$ depending only on d, n, U, and φ such that*

$$\left\| f - \sum_{k\in\mathbb{Z}^d} \sum_{l} b_l \frac{m^j}{\text{mes}\, U} \int\limits_{M^{-j}(U-l)} f(-M^{-j}k+t)\,dt\, \varphi(M^j\cdot+k) \right\|_p \le C\omega_n(f, \|M^{-j}\|)_p$$

where C does not depend on f and j.

Example 3. Let $d=2$, $U=B_1$, and let $\varphi(x)=R_\delta(x)$ be given by the Bochner-Riesz type kernel

$$R_\delta(x) := \frac{\Gamma(1+\delta)}{\pi^\delta}\frac{J_{d/2+\delta}(2\pi|x|)}{|x|^{d/2+\delta}}$$

Then

$$\left\| f - \sum_{k\in\mathbb{Z}^2}\sum_{0\le l_1,l_2\le 3}\frac{b_{l_1,l_2}m^j}{\pi}\int\limits_{M^{-j}(B_1-(l_1,l_2))} f(-M^{-j}k+t)\,dt\,R_\delta(M^j\cdot+k)\right\|_p \le C\omega_4(f,\|M^{-j}\|)_p$$

where

$$b_{0,0}=1-\frac{2\delta-\pi^2}{2\pi^2},\quad b_{1,0}=b_{0,1}=\frac{5(2\delta-\pi^2)}{8\pi^2}$$
$$b_{2,0}=b_{0,2}=-\frac{2\delta-\pi^2}{2\pi^2},\quad b_{3,0}=b_{0,3}=\frac{2\delta-\pi^2}{8\pi^2}$$

and

$$b_{1,1}=b_{1,2}=b_{2,1}=0$$

Thank you for attention!