On approximation processes defined by the cosine operator function in a Banach space

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Introduction

The aim of this presentation is to introduce an abstract framework of certain approximation processes using a cosine operator functions concept.

Historical roots of these processes go back to W.W. Rogosinski, 1926, who proved that the arithmetical mean of shifted Fourier partial sums converges uniformly to a given 2π-periodic continuous functions.

In notations: for $f \in C_{2\pi}$ the Fourier partial sums

$$S_n(f,x) = \frac{a_0}{2} + \sum_{k=1}^{n} a_k \cos kx + b_k \sin kx$$

define the Rogosinski means by

$$R_n(f,x) := \frac{1}{2}\left(S_n(f,x+\frac{\pi}{2(n+1)}) + S_n(f,x-\frac{\pi}{2(n+1)})\right).$$

X - be an arbitrary (real or complex) Banach space.

$[X]$ - be the Banach algebra of all bounded linear operators $U : X \to X$

$X \supset A_\sigma$ - be a dense family of subsets, meaning that for every $f \in X$ there exists a family $\{g_\sigma\}_{\sigma>0}, g_\sigma \in A_\sigma$ such that $\lim_{\sigma\to\infty} \|f - g_\sigma\| = 0$.

$S : A_\sigma \to A_\sigma$ - be a linear projection operator

Definition

A **cosine operator function** $T_h \in [X]$ $(h \geq 0)$ is defined by the properties:

(i) $T_0 = I$(identity operator),

(ii) $T_{h_1} \cdot T_{h_2} = \frac{1}{2}(T_{h_1+h_2} + T_{|h_1-h_2|})$,

(iii) $\|T_h f\| \leq T\|f\|$, $0 < T-$ not depending on $h > 0$.

Let $\tau_h \in [X],\ h \in \mathbb{R}$, be a **translation operator**, defined by the properties

(i) $\tau_0 = I$,

(ii) $\tau_{h_1} \cdot \tau_{h_2} = \tau_{h_1+h_2}$,

(iii) $\|\tau_h f\| \le T\|f\|, 0 < T-$ not depending on $h \in \mathbb{R}$.

Then $T_h := \frac{1}{2}(\tau_h + \tau_{-h}),\ h \ge 0$, is a cosine operator function.

It means that if we can define a translation operator, then we have also the cosine operator function.

A non-trivial cosine operator function related with the Fourier-Chebyshev series: $x \in [-1,1],\ 0 \le h \le \pi$,

$$T_h^C(f,x) := \frac{1}{2}\left\{f(x\cos h + \sqrt{1-x^2}\sin h) + f(x\cos h - \sqrt{1-x^2}\sin h)\right\}.$$

But for some spaces we cannot define the translation operator $\tau_h \in [X],\ h \in \mathbb{R}$, nevertheless the cosine operator function does exist.

An **example**: $X = C_{2\pi}^{-}$ - space of π-symmetric and 4π-periodic continuous functions, i.e. $f(\pi - x) = f(\pi + x)$ and $f(4\pi + x) = f(x)$ for all $x \in \mathbb{R}$.

For example, the functions $y = \sin(k - \frac{1}{2})x$, $k \in \mathbb{N}$ are in space $C_{2\pi}^{-}$.

Here $\tau_h \left(\sin\left(\frac{1}{2}\circ\right), x\right) = \sin\frac{1}{2}(x + h) \notin C_{2\pi}^{-}$ for some $h \in \mathbb{R}$, but $T_h f \in C_{2\pi}^{-}$, where $T_h := \frac{1}{2}(\tau_h + \tau_{-h})$ and τ_h is the ordinary translation operator.

Example = Trigonometric approximation

$X = C_{2\pi}$, the space of 2π-periodic continuous functions,

with dense subspaces $A_n \subset C_{2\pi}$ consisting of trigonometric polynomials of degree not exceeding n;

Fourier partial sum operators $S_n : A_n \to A_n$ are here the linear projection operators.

In this setting the trigonometric Rogosinski means have the shape

$$R_n(f,x) = T_{\frac{\pi}{2(n+1)}} S_n(f,x).$$

Important: The projection operators, hence the Rogosinski means, are defined on the whole space $C_{2\pi}$.

Example = Shannon sampling operators

$X = C(\mathbb{R})$, the space of uniformly continuous and bounded functions on $\mathbb{R}$ with dense subspaces $B^\infty_\sigma \subset C(\mathbb{R})$ consisting of bounded

functions on $\mathbb{R}$, which are entire functions $f(z)$ $(z \in \mathbb{C})$ of exponential type σ, i.e. $|f(z)| \leq e^{\sigma|y|}\|f\|_C \quad (z = x + iy \in \mathbb{C})$.

Linear **projection operator** in this case is the classical Whittaker-Kotel'nikov-Shannon operator, for $g \in B^\infty_\sigma, \sigma < \pi w$ defined by

$$(S^{\text{sinc}}_w g)(t) := \sum_{k=-\infty}^{\infty} g\left(\frac{k}{w}\right) \operatorname{sinc}(wt - k),$$

where the kernel function $\operatorname{sinc}(t) := \frac{\sin \pi t}{\pi t}$.

Example = Shannon sampling operators continued

To be the projection operator is a statement of famous **Whittaker-Kotel'nikov-Shannon theorem**:
if $g \in B_\sigma^\infty, \sigma < \pi w$, then

$$(S_w^{\text{sinc}} g)(t) = g(t).$$

Important: The projection operators $S_w^{\text{sinc}} : B_\sigma^\infty \to B_\sigma^\infty$ are defined only on dense subspaces $B_\sigma^\infty \subset C(\mathbb{R})$.

Theorem

Extension Theorem *([Kantorovich-Akilov], Ch.V, Sect. 8.2, 8.3) Let $A_\sigma \subset X$ be a family of dense subsets of a Banach space X and $\widetilde{B} : A_\sigma \to X$ is a bounded linear operator with the operator norm $\|\widetilde{B}\|$. Then $\widetilde{B}$ has a bounded linear extension $B : X \to X$ with $\|B\| = \|\widetilde{B}\|$. For $f \in X$ the operator $B \in [X]$ is defined by $Bf = \lim_{\sigma\to\infty} \widetilde{B} g_\sigma$, where $\{g_\sigma\}_{\sigma>0} \subset A_\sigma$ is an arbitrary family with $f = \lim_{\sigma\to\infty} g_\sigma$.*

Rogosinski-type operators

Definition

Let $A_\sigma \subset X$ be a dense family of subsets and $S_\sigma : A_\sigma \to A_\sigma$ be a linear projection operator, moreover, $T_h \in [X]$ be a cosine operator function. The Rogosinski-type operator $R_{\sigma,h,a} : X \to X$ is defined as an extension of $\widetilde{R}_{\sigma,h,a} : A_\sigma \to X$, which is defined by

$$\widetilde{R}_{\sigma,h,a} g := a T_h(S_\sigma g) + (1-a)\, T_{3h}(S_\sigma g) \quad (h \geq 0, a \in \mathbb{R}).$$

Example. Let $A_n \subset C_{2\pi}$ be the set of trigonometric polynomials and $S_n : A_n \to A_n$ be the Fourier partial sums operators. Then the given Definition for $a = 1$ leads to the historical Rogosinski means.

Remark. The Fourier partial sums operators are defined on the whole space $X = C_{2\pi}$, but this is not true for the Whittaker-Kotel'nikov-Shannon operator. This is the reason why we first define our approximation processes on a dense subspace.

Order of approximation by Rogosinski-type operators

To measure the order of approximation, two traditional quantities are known - the **best approximation** and the **modulus of continuity**.

Definition

The **best approximation** of $f \in X$ by elements of A is defined by

$$E_A(f) := \inf_{g \in A} \|f - g\|.$$

Remark We often may suppose that there exists an element $g* \in A$ of best approximation, i.e. $E_A(f) = \|f - g*\|$.

Definition

The **modulus of continuity** of order k is defined via the cosine operator function by

$$\omega_k(f, \delta) := \sup_{0 \leq h \leqslant \delta} \|(T_h - I)^k f\|, \; k \in \mathbb{N}.$$

Remark

Let $\tau_h \in [X],\ h \in \mathbb{R}$, be a translation operator, defined by the properties

(i) $\tau_0 = I$,

(ii) $\tau_{h_1} \cdot \tau_{h_2} = \tau_{h_1+h_2}$,

(iii) $\|\tau_h f\| \leq T\|f\|, 0 < T -$ not depending on $h \in \mathbb{R}$.

Since $T_h := \frac{1}{2}(\tau_h + \tau_{-h}),\ h \geq 0$ is a cosine operator and $T_h - I = \frac{1}{2}\left(\tau_{h/2} - \tau_{-h/2}\right)^2$, then the modulus of continuity, defined above, can be represented by

$$\omega_k(f,\delta) = \frac{1}{2^k}\widetilde{\omega}_{2k}(f,\delta),$$

where

$$\widetilde{\omega}_k(f,\delta) := \sup_{0\leq h\leq\delta} \left\|\left(\tau_{h/2} - \tau_{-h/2}\right)^k f\right\|,\ k \in \mathbb{N}.$$

Theorem

For every $f \in X$, $a \in \mathbb{R}$ for the Rogosinski-type operators $R_{\sigma,h,a} : X \to X$ there holds

$$\begin{aligned} \|R_{\sigma,h,a}f - f\| &\leq \left(\|R_{\sigma,h,a}\|_{[X]} + |a|T + |1-a|T\right) E_{A_\sigma}(f) \\ &+ |a|\omega(f,h) + |1-a|\omega(f,3h). \end{aligned}$$

The importance of parameters $a \in \mathbb{R}$ will be explained by the next theorem. It appears that the particular value $a = \frac{9}{8}$ yields a better order of approximation. **Attention:** The next one is not a corollary of the previous one.

Theorem

Denote $\overline{R}_{\sigma,h} = R_{\sigma,h,9/8}$. Then we have

$$\|\overline{R}_{\sigma,h}f - f\| \leq \left(\|\overline{R}_{\sigma,h}\|_{[X]} + \frac{5}{4}T\right) E_{A_\sigma}(f) + \frac{3}{2}\omega_2(f,h) + \frac{1}{2}\omega_3(f,h).$$

Blackman-Harris-type operators

Definition

Let $A_\sigma \subset X$ be a dense family of subsets and $S_\sigma : A_\sigma \to A_\sigma$ be a linear projection operator, moreover, $T_h \in [X]$ be a cosine operator function. The Blackman-Harris-type operator $C_{\sigma,h,c} : X \to X$ is defined as an extension of $\widetilde{C}_{\sigma,h,c} : A_\sigma \to X$, which is defined by

$$\widetilde{C}_{\sigma,h,c}g := \sum_{k=0}^{m} c_k T_{kh}(S_\sigma g), \quad h \geq 0, \quad c = (c_0, ..., c_m) \in \mathbb{R}^{m+1}$$

with

$$\sum_{k=0}^{m} c_k = 1.$$

The Blackman-Harris-type operators in Trigonometric Approximation reduce to the φ-means or φ-summation methods generated by a single function

$$\varphi_c(t) = \sum_{k=0}^{m} c_k \cos kt$$

with the corresponding operators

$$C_{n,h,c}(f,x) := \frac{a_0}{2} + \sum_{k=1}^{n} \varphi_c(kh)\,(a_k \cos kx + b_k \sin kx)\,.$$

The choice of the increment h depends on the concrete operator, e.g., for the trigonometric Rogosinski operator $h = \pi/(2(n+1))$.

Theorem

For every $f \in X$, $c = (c_0, ..., c_m) \in \mathbb{R}^{m+1}$ *for the Blackman-Harris-type operator* $C_{\sigma,h,c} : X \to X$ *there holds*

$$\begin{aligned}\|C_{\sigma,h,c}f - f\| \ &\leq \ \left(\|C_{\sigma,h,c}\|_{[X]} + |c_0| + T\sum_{k=1}^{m}|c_k| \right) E_{A_\sigma}(f) \\ &+ \ \sum_{k=1}^{m}|c_k|\omega(f, kh).\end{aligned}$$

Remark The higher order of approximation via $\omega_l(f, h)$ depends, in fact, due to an isomorphism between cosines and the cosine operator functions, on a trigonometric equation in form

$$\sum_{k=0}^{m} c_k \cos kx = (\cos x - 1)^l \sum_{k=0}^{m-l} d_k \cos kx, \ \ m \geq l, \ \ d_k \in \mathbb{R} \ (k = 0, ..., m-l)$$

A similar abstract framework can be applied for the integral operators

$$K_{\sigma,h,c}f := \int_0^1 C_{\sigma,uh,c} f du,$$

where the integral here is the Riemann integral in a Banach space (see [L. Schwartz, Cours d'analyse (Vol. 1) Analyse Mathématique Cours, 1967])

Example In Shannon sampling theory the Lanczos operators $L_w : C(\mathbb{R}) \to C(\mathbb{R})$ can be defined by

$$L_w(f,t) = \int_0^1 T_{u/(2w)}(R_w f, t) du,$$

where the Rogosinski-type sampling operator is defined by

$$R_w(f,t) := \sum_{k=-\infty}^{\infty} f(\frac{k}{w}) r(wt-k)$$

with the kernel function $r \in L^1(\mathbb{R}), r(t) = \frac{\cos \pi t}{2\pi(1/4 - t^2)}$.

Concluding remarks

Positive moments

- Formally the abstract Blackman-Harris – type operators possess good approximation properties, since the modulus of continuity and operators both are defined by a cosine operator function; these have also practical applications, e.g. in signal and imaging processes.
- The given abstract setting has many concrete applications: Shannon sampling series, singular integrals of Fourier transform, trigonometric Fourier approximation, Fourier-Chebyshev series, Fourier-Walsh series (keeping in mind the dyadic translation) etc.

A negative moment

- In abstract setting we cannot say very much about the uniform boundedness of this operators family, but in concrete function spaces we have computed exact operator norms, also important in applications.

About boundedness:

As the Rogosinski-type operators

$$\widetilde{R}_{\sigma,h,a}g := aT_h(S_\sigma g) + (1-a)\,T_{3h}(S_\sigma g)$$

possess the representation

$$U_a = aV + W,\ \ V, W \in [X],$$

then for every $a, a_0, a_1 \in \mathbb{R}$ we get

$$(a_1 - a_0)U_a = (a_1 - a)U_{a_0} + (a - a_0)U_{a_1}.$$

This equality shows that for boundedness of $\|U_a\|_{[X]}$ for arbitrary $a \in \mathbb{R}$ it is enough to know the boundedness of $\|U_{a_0}\|_{[X]}$ and $\|U_{a_1}\|_{[X]}$ for two specific values $a_0, a_1 \in \mathbb{R}$.

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KÖSZÖNÖM SZÉPEN !

THANK YOU !