Approximation problems in the variable exponent Lebesgue spaces

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- In this talk we discuss the approximation problems in the variable exponent Lebesgue spaces.

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- Constructive characterization problems

2 INTRODUCTION

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- Orlicz W. : *Über konjugierte Exponentenfolgen*, Studia Math. 3, (1931), pp. 200-212.

- Interest in the variable exponent Lebesgue spaces has increased since 1990s, because of their use in the different applications problems in mechanic, especially in fluid dynamic for the modelling of electrorheological fluids. These are fluids whose viscosity chances (often dramatically) when exposed to an electric field. The variable exponent Lebesgue spaces are also used in the study of image processing and some physical problems.

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- Diening L., Harjulehto P., Hästö P., Michael Ruzicka M.: *Lebesgue and Sobolev Spaces with Variable Exponents*, Springer, Heidelberg Dordrecht London New York(2011).

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- Sharapudinov I. I. : *Some questions of approximation theory in the Lebesgue spaces with variable exponent:*
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- Meanwhile, the approximation problems in these spaces, especially in the complex plane were not investigated sufficiently wide.

- Let $\mathbb{T} := [0, 2\pi]$ and let $p(\cdot) : \mathbb{T} \to [0, \infty)$ be a Lebesgue measurable 2π periodic function such that

$$1 \leq p_{-} := ess \inf_{x \in \mathbb{T}} p(x) \leq ess \sup_{x \in \mathbb{T}} p(x) := p^{+} < \infty.$$

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- In addition to this requirement if

$$|p(x) - p(y)| \ln \frac{2\pi}{|x-y|} \leq d, \ \forall x, y \in [0, 2\pi]$$

with a positive constant d, then we say that $p(\cdot) \in \mathcal{P}(\mathbb{T})$. We also define $P_0(\mathbb{T}) := \{p(\cdot) \in \mathcal{P}(\mathbb{T}) : p_{-} > 1\}$.

- The variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ is defined as the set of all Lebesgue measurable 2π periodic functions f such that

$$\rho_{p(\cdot)}(f) := \int_0^{2\pi} |f(x)|^{p(x)}\, dx < \infty.$$

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- Equipped with the norm

$$\|f\|_{p(\cdot)} = \inf\left\{\lambda > 0 : \rho_{p(\cdot)}(f/\lambda) \leq 1\right\}$$

it becomes a Banach space.

- One of the main problem observed in the investigations on the approximation theory is the correct definition of the modulus of smoothness. It is a fact that $L^{p(\cdot)}(\mathbb{T})$ is noninvariant with respect to the usual shift operator $f(\cdot + h)$, in general.

- Nevertheless, the Steklov mean value operator

$$\sigma_h(f) := \frac{1}{h}\int_0^h f(x+t)\,dt, \quad h > 0$$

is bounded in $L^{p(\cdot)}(\mathbb{T})$. See,

Diening L., Růžiĉka M. : *Calderon-Zigmund operators on generalized Lebesgue spaces $L^{p(x)}$ and problems related to fluid dynamic*, J. Reine Angew. Math., Vol. 563, (2003), pp. 197-220).

- By using this boundedness was constructed by us the *first order modulus of smoothness*

$$\Omega_{p(\cdot)}(f,\delta) := \sup_{0<h\leq\delta} \left\| \frac{1}{h} \int_0^h |f(\cdot) - f(\cdot + t)| \, dt \right\|_{p(\cdot)}$$

and was obtained the direct theorem of approximation theory in $L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$, and also some results on the approximation by the Nörlund means of Fourier series in $L^{p(\cdot)}(\mathbb{T})$. See:

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- Guven A. and Israfilov D. M. : *Trigonometric Approximation in Generalized Lebesgue Spaces* $L^{p(x)}$, Journal of Math. Inequalities, Vol. 4, No: 2, (2010), pp. 285-299.

- Similar results under the condition of $p(\cdot) \in \mathcal{P}_0(\mathbb{T})$ using some other modulus of smoothness were stated or proved in the papers:

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- Akgun R. and Kokilashvili V. M. : *The refined direct and converse inequalities of trigonometric approximation in weighted variable exponent Lebesgue spaces*, Georgian Math. Journal, 18, (2011), pp. 399-423.

- In the more general case, i.e. in the case of $p(\cdot) \in \mathcal{P}(\mathbb{T}) \supset \mathcal{P}_0(\mathbb{T})$ using the modulus

$$\Omega(f,\delta)_{p(\cdot)} := \sup_{0<h\leq\delta} \left\| \frac{1}{h} \int_0^h [f(\cdot) - f(\cdot + t)]\, dt \right\|_{p(\cdot)}$$

which is more sensitive than $\Omega_{p(\cdot)}(f,\delta)$, the direct and inverse theorems were proved by Sharapudinov in the above cited his monograph.

- In term of $\Omega(f,\delta)_{p(\cdot)}$ with $p(\cdot) \in \mathcal{P}(\mathbb{T})$, one general inverse theorem which generalizes the inverse theorem obtained by Sharapudinov was proved in the work:

- In term of $\Omega\left(f,\delta\right)_{p(\cdot)}$ with $p\left(\cdot\right)\in\mathcal{P}\left(\mathbb{T}\right)$, one general inverse theorem which generalizes the inverse theorem obtained by Sharapudinov was proved in the work:
- Israfilov D. M. and Testici A. : *Approximation in Smirnov Classes with Variable Exponent*, Complex Variables and Elliptic Equations, Vol. 60, No: 9, (2015), pp.1243-1253.

3 NEW RESULTS

- We discuss some results obtained by us on the approximation problems in $L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$, in the term of *the r*th $(r = 1, 2, ...)$ *modulus of smoothness* $\Omega_r(f, \delta)_{p(\cdot)}$.

Definition (1)

We define the r-th modulus of smoothness as

$$\Omega_r(f, \delta)_{p(\cdot)} := \sup_{0<h\leq\delta} \left\| \frac{1}{h} \int_0^h \Delta_t^r f dt \right\|_{p(\cdot)}, \quad \delta > 0.$$

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- *Let* $f \in L^{p(\cdot)}(\mathbb{T})$ *with* $p(\cdot) \in \mathcal{P}(\mathbb{T})$ *and let*

$$\Delta_t^r f(x) := \sum_{s=0}^{r} (-1)^{r+s} \binom{r}{s} f(x + st) \ , \qquad r = 1, 2, \ldots \ .$$

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- For $f \in L^{p(\cdot)}(\mathbb{T})$ we define the best approximation number

$$E_n(f)_{p(\cdot)} := \inf \left\{ \|f - T_n\|_{p(\cdot)} : T_n \in \Pi_n \right\}$$

in the class Π_n of the trigonometric polynomials of degree not exceeding n.

- Throughout this talk by $c(\cdot)$, $c(\cdot,\cdot)$, $c_1(\cdot,\cdot)$, $c_2(\cdot,\cdot)$,... we denote the constants (which can be different in different relations) depending only on the parameters given in the corresponding brackets.

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- The main direct and inverse results obtained in the spaces $L^{p(\cdot)}([0,2\pi])$ are following.

Theorem (1)

Let $p(\cdot) \in \mathcal{P}(\mathbb{T})$, $r \in \mathbb{N}$. Then there exists a positive constant $c(p,r)$ such that for every $f \in L^{p(\cdot)}(\mathbb{T})$ and $n \in \mathbb{N}$ the inequality

$$E_n(f)_{p(\cdot)} \leq c(p,r)\Omega_r(f,1/n)_{p(\cdot)}$$

holds.

Theorem (2)

Let $p(\cdot) \in \mathcal{P}(\mathbb{T})$, $r \in \mathbb{N}$. *Then there exists a positive constant* $c(p,r)$ *such that for every* $f \in L^{p(\cdot)}(\mathbb{T})$ *and* $n \in \mathbb{N}$ *the inequality*

$$\Omega_r(f, 1/n)_{p(\cdot)} \leq \frac{c(p,r)}{n^r} \sum_{k=0}^{n} (k+1)^{r-1} E_k(f)_{p(\cdot)}$$

holds.

- Denoting by

$$W_k^{p(\cdot)}(\mathbb{T}) := \left\{ f : f^{(k-1)} \text{ is absolutely continuous and } f^{(k)} \in L^{p(\cdot)}(\mathbb{T}) \right\}$$

$k = 1, 2, ...,$ the variable exponent Sobolev space and combining *Theorem 1* with the estimation

$$E_n(f)_{p(\cdot)} \leq \frac{c(p)}{n^k} E_n\left(f^{(k)}\right)_{p(\cdot)},$$

which can be deduced from Sharapudinov's work : *On Direct and Inverse Theorems of Approximation Theory In Variable Lebesgue Space And Sobolev Spaces, Azerbaijan Journal of Math., Vol. 4, No 1, (2014), pp. 55-72.*, we have

Corollary (1)

Let $p(\cdot) \in \mathcal{P}(\mathbb{T})$, $k \in \mathbb{N}$. *Then there exists a positive constant* $c(p,r)$ *such that for every* $f \in W_k^{p(\cdot)}(\mathbb{T})$ *and,* $n \in \mathbb{N}$ *the following inequality holds*

$$E_n(f)_{p(\cdot)} \leq \frac{c(p,r)}{n^k} \Omega_r \left(f^{(k)}, 1/n \right)_{p(\cdot)}.$$

On the other hand, *Theorem 2* implies

Corollary (2)

If $E_n(f)_{p(\cdot)} = \mathcal{O}(n^{-\alpha})$, $\alpha > 0$, *then under the conditions of Theorem 2*

$$\Omega_r(f,\delta)_{p(\cdot)} = \begin{cases} \mathcal{O}(\delta^{\alpha}) & ,r > \alpha \\ \mathcal{O}(\delta^{\alpha}\log(1/\delta)) & ,r = \alpha \\ \mathcal{O}(\delta^{r}) & ,r < \alpha. \end{cases}$$

- Hence, if we define a generalized Lipschitz class $Lip_{\alpha}^{p(\cdot)}(\mathbb{T})$ for $\alpha > 0$ and $r := [\alpha] + 1$ ($[\alpha]$ is the integer part of α) as

$$Lip_{\alpha}^{p(\cdot)}(\mathbb{T}) := \left\{ f \in L^{p(\cdot)}(\mathbb{T}) : \Omega_r(f,\delta)_{p(\cdot)} = \mathcal{O}(\delta^{\alpha}), \quad \delta > 0 \right\},$$

then we have

Corollary (3)

If $E_n(f)_{p(\cdot)} = \mathcal{O}(n^{-\alpha})$, $\alpha > 0$, then under the conditions of Theorem 2, $f \in Lip_{\alpha}^{p(\cdot)}(\mathbb{T})$.

On the other hand, from *Theorem 1* we also get

Corollary (4)

If $f \in Lip_{\alpha}^{p(\cdot)}(\mathbb{T})$ with $p(\cdot) \in \mathcal{P}(\mathbb{T})$ and for some $\alpha > 0$, then $E_n(f)_{p(\cdot)} = \mathcal{O}(n^{-\alpha})$.

- Now *Corollaries 3 and 4* imply

Theorem (3)

Let $f \in L^{p(\cdot)}(\mathbb{T})$, $p(\cdot) \in \mathcal{P}(\mathbb{T})$, *and let* $\alpha > 0$. *The following statements are equivalent:*

$$\begin{aligned} i) f &\in Lip_{\alpha}^{p(\cdot)}(\mathbb{T}), \\ ii) E_n(f)_{p(\cdot)} &= \mathcal{O}\left(n^{-\alpha}\right), n \in \mathbb{N}. \end{aligned}$$

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- Note that when $p(\cdot) =$*constant* these results coincide with the classical results, proved by different authors.

4 NEW RESULT IN THE COMPLEX DOMAINS

- Let $G \subset \mathbb{C}$ be a finite domain in the complex plane, bounded by a rectifiable Jordan curve Γ and let $G^{-} := Ext\ \Gamma$. Let also $\mathbb{T} := \{w \in \mathbb{C} : |w| = 1\}$, $D := Int\ \mathbb{T}$ and $D^{-} := Ext\ \mathbb{T}$.

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Definition (2)

The variable exponent Lebesgue spaces $L^{p(\cdot)}(\Gamma)$ for a given nonnegative Lebesgue measurable variable exponent $p(z) \geq 1$ on Γ we define as the set of Lebesgue measurable functions f, such that

$$\int_{\Gamma} |f(z)|^{p(z)} |dz| < \infty.$$

- Equipped with the norm

$$\|f\|_{L^{p(\cdot)}(\Gamma)} := \inf\left\{\lambda \geq 0 : \int\limits_{\Gamma} \left|\frac{f(z)}{\lambda}\right|^{p(z)} |dz| \leq 1\right\} < \infty$$

$L^{p(\cdot)}(\Gamma)$ becomes a Banach spaces.

- In the case of $\Gamma := \mathbb{T}$ we obtain the variable exponent Lebesgue space $L^{p(\cdot)}(\mathbb{T})$ with the norm

$$\|f\|_{L^{p(\cdot)}(\mathbb{T})} := \inf\left\{\lambda \geq 0 : \int\limits_0^{2\pi} \left|\frac{f(e^{it})}{\lambda}\right|^{p(e^{it})} |dt| \leq 1\right\} =: \|f\|_{L^{p(\cdot)}([0,2\pi])}.$$

- Let $E^1(G)$ be the classical Smirnov class of analytic functions in G. The Smirnov classes in detail were investigated in the monograph:

Definition (3)

Let $p(\cdot) : \Gamma \to [1, \infty)$ be a Lebesgue measurable function. The set

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- Goluzin G. M. : *Geometric Theory of Functions of a Complex Variable*. Translation of Mathematical Monographs, Vol. 26, AMS 1969.

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- In particular if $G := D$, then we have variable exponent Hardy spaces $H^{p(\cdot)}(D)$.
  Let Γ be a Jordan rectifiable curve in the complex plane $\mathbb{C}$ and let $p(\cdot) : \Gamma \to R^+$ be a measurable function defined on Γ such that

$$1 \leq p_- := ess \inf_{z\in\Gamma} p(z) \leq ess \sup_{z\in\Gamma} p(z) := p^+ < \infty. \qquad (1)$$

Definition (4)

We say that $p(\cdot) \in \mathcal{P}(\Gamma)$, if $p(\cdot)$ satisfies the conditions (1) and

$$|p(z_1) - p(z_2)| \ln \frac{|\Gamma|}{|z_1 - z_2|} \leq c, \qquad \forall z_1, z_2 \in \Gamma$$

with a positive constant c, where $|\Gamma|$ is the Lebesgue measure of Γ.

- If $p(\cdot) \in \mathcal{P}(\Gamma)$ with $p_- > 1$, then we say that $p(\cdot) \in \mathcal{P}_0(\Gamma)$.

- Let Γ be a smooth Jordan curve and let $\theta(s)$ be the angle between the tangent and the positive real axis expressed as a function of arclength s. If Γ has a modulus of continuity $\omega(\theta, s)$, satisfying the Dini-smooth condition

$$\int_0^\delta \omega(\theta, s) / s \, ds < \infty, \quad \delta > 0,$$

then we say that Γ is a Dini smooth curve and the set of Dini-smooth curves we denote by $\mathcal{D}$.

- By φ we denote the conformal mapping of G^{-} onto D^{-}, normalized by the conditions

$$\varphi(\infty) = \infty \text{ and } \lim_{z \to \infty} \varphi(z)/z > 0.$$

Let ψ be the inverse mapping of φ.

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- The mappings φ and ψ have continuous extensions to Γ and $\mathbb{T}$, respectively. Their derivatives φ' and ψ' have definite nontangential limit values a.e. on Γ and $\mathbb{T}$, and the limit functions are integrable with respect to Lebesgue measure on Γ and $\mathbb{T}$, respectively.

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- For a given function $f \in L^{p(\cdot)}(\Gamma)$ with $p \in \mathcal{P}(\Gamma)$ we set

$$f_0(w) := f[\psi(w)]$$

$$p_0(w) := p(\psi(w)).$$

- If $\Gamma \in \mathcal{D}$, then as follows from *Warschawski's* works, there are the positive constants $c_i > 0, i = 1, 2, 3, 4$ such that

$$
\begin{aligned}
0 &< c_1 \leq \left|\psi^{'}(w)\right| \leq c_2 < \infty, \\
0 &< c_3 \leq \left|\varphi^{'}(z)\right| \leq c_4 < \infty,
\end{aligned}
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  a.e. on $\mathbb{T}$ and on Γ, respectively.
- Therefore if $\Gamma \in \mathcal{D}$, then

$$f \in L^{p(\cdot)}(\Gamma) \Leftrightarrow f_0 \in L^{p_0(\cdot)}(\mathbb{T}).$$

- Moreover,

$$\|f_0\|_{L^{p_0}(\mathbb{T})} \leq c_9 \|f\|_{L^{p(\cdot)}(\Gamma)} \leq c_{10} \|f_0\|_{L^{p_0(\cdot)}(\mathbb{T})}$$

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- It is also clear that if $\Gamma \in \mathcal{D}$, then

$$p_0(\cdot) \in \mathcal{P}(\mathbb{T}) \Leftrightarrow p(\cdot) \in \mathcal{P}(\Gamma).$$

- Moreover,

$$\left\| f_0 \right\|_{L^{p_0}(\mathbb{T})} \leq c_9 \left\| f \right\|_{L^{p(\cdot)}(\Gamma)} \leq c_{10} \left\| f_0 \right\|_{L^{p_0(\cdot)}(\mathbb{T})}$$

- It is also clear that if $\Gamma \in \mathcal{D}$, then

$$p_0(\cdot) \in \mathcal{P}(\mathbb{T}) \Leftrightarrow p(\cdot) \in \mathcal{P}(\Gamma).$$

- For a given function $f \in L^{p(\cdot)}(\Gamma)$ we define the Cauchy type integral

$$f_0^+(w) := \frac{1}{2\pi i} \int_{\mathbb{T}} \frac{f_0(\tau)}{\tau - w} d\tau$$

which are analytic in D.

- For a given $f \in L^{p(\cdot)}(\mathbb{T})$, defining the mean value function on the unit circle $\mathbb{T}$ as

$$\sigma_h f(w) := \frac{1}{h} \int_0^h f\left(we^{it}\right) dt, \qquad w \in \mathbb{T}$$

we obtain the following modification of the modulus of smoothness of f on $\mathbb{T}$:

$$\Omega(f, \delta)_{\mathbb{T}, p(\cdot)} := \sup_{0<h\leq\delta} \left\| f(w) - \sigma_h f(w) \right\|_{L^{p(\cdot)}(\mathbb{T})}.$$

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- If $f \in E^{p(\cdot)}(G)$, then we define the modulus of smoothness

$$\Omega(f, \delta)_{G,p(\cdot)} := \Omega\left(f_0^+, \delta\right)_{\mathbb{T},p_0(\cdot)}, \qquad \delta > 0$$

for f.

- The best approximation number of $f \in E^{p(\cdot)}(G)$ is defined by

$$E_n(f)_{G,p(\cdot)} := \inf \left\{ \|f - P_n\|_{L^{p(\cdot)}(\Gamma)} : P_n \in \Pi_n \right\},$$

where Π_n is the class of algebraic polynomials of degree not exceeding n.

- For simplicity we are formulate the results only for the first modulus. For the higher moduli the appropriate results also are true.

Theorem (4)

Let $\Gamma \in \mathcal{D}$. *If* $f \in E^{p(\cdot)}(G)$ *with* $p(\cdot) \in \mathcal{P}_0(\Gamma)$, *then*

$$E_n(f)_{G,p(\cdot)} \leq c(p)\ \Omega(f, 1/n)_{G,p(\cdot)}$$

with a constant $c > 0$ *independent of* n.

Theorem (5)

Let $\Gamma \in \mathcal{D}$. *If* $f \in E^{p(\cdot)}(G)$ *with* $p(\cdot) \in \mathcal{P}_0(\Gamma)$, *then*

$$\Omega(f, 1/n)_{G,p(\cdot)} \leq \frac{c(p)}{n} \sum_{v=0}^{n} E_v(f)_{G,p(\cdot)} \qquad n = 1, 2, ...,$$

- For simplicity we are formulate the results only for the first modulus. For the higher moduli the appropriate results also are true.
- Then the direct and inverse results obtained in the classes $E^{p(\cdot)}(G)$ can be formulated as following:

Theorem (4)

Let $\Gamma \in \mathcal{D}$. *If* $f \in E^{p(\cdot)}(G)$ *with* $p(\cdot) \in \mathcal{P}_0(\Gamma)$, *then*

$$E_n(f)_{G,p(\cdot)} \leq c(p)\ \Omega(f, 1/n)_{G,p(\cdot)}$$

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Theorem (5)

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$$\Omega(f, 1/n)_{G,p(\cdot)} \leq \frac{c(p)}{n} \sum_{v=0}^{n} E_v(f)_{G,p(\cdot)} \qquad n = 1, 2, ...,$$

- Defining the generalized Lipschitz class $Lip_{p(\cdot)}\left(G,\alpha\right)$ with $\alpha\in\left(0,1\right)$ by

$$Lip_{p(\cdot)}\left(G,\alpha\right):=\left\{f\in E^{p(\cdot)}(G):\Omega\left(f,\delta\right)_{G,p(\cdot)}=\mathcal{O}\left(\delta^{\alpha}\right),\ \ \delta>0\right\},$$

from Theorem (5) after simple computations we obtain:

Corollary (5)

Let $\Gamma\in\mathcal{D}$ and $p(\cdot)\in\mathcal{P}_0(\Gamma)$. If $E_n\left(f\right)_{G,p(\cdot)}=O\left(n^{-\alpha}\right)$ with $\alpha\in\left(0,1\right)$, then $f\in Lip_{p(\cdot)}\left(G,\alpha\right)$.

Corollary (6)

If $f\in Lip_{p(\cdot)}\left(G,\alpha\right)$ with $p(\cdot)\in\mathcal{P}_0(\Gamma)$ and $\alpha\in\left(0,1\right)$, then

$$E_n\left(f\right)_{G,p(\cdot)}=O\left(n^{-\alpha}\right).$$

- Defining the generalized Lipschitz class $Lip_{p(\cdot)}(G,\alpha)$ with $\alpha \in (0,1)$ by

$$Lip_{p(\cdot)}(G,\alpha) := \left\{ f \in E^{p(\cdot)}(G) : \Omega(f,\delta)_{G,p(\cdot)} = \mathcal{O}(\delta^{\alpha}),\ \delta > 0 \right\},$$

from Theorem (5) after simple computations we obtain:

Corollary (5)

Let $\Gamma \in \mathcal{D}$ and $p(\cdot) \in \mathcal{P}_0(\Gamma)$. If $E_n(f)_{G,p(\cdot)} = O(n^{-\alpha})$ with $\alpha \in (0,1)$, then $f \in Lip_{p(\cdot)}(G,\alpha)$.

- At the same time Theorem (4) implies

Corollary (6)

If $f \in Lip_{p(\cdot)}(G,\alpha)$ with $p(\cdot) \in \mathcal{P}_0(\Gamma)$ and $\alpha \in (0,1)$, then

$$E_n(f)_{G,p(\cdot)} = O(n^{-\alpha}).$$

- **The corollaries (5) and (6)** imply the following constructive characterization of $Lip_{p(\cdot)}(G, \alpha)$:

Theorem (6)

Let $\Gamma \in \mathcal{D}$ *and* $p(\cdot) \in \mathcal{P}_0(\Gamma)$*, and let* $\alpha \in (0, 1)$*. The following statements are equivalent:*

$$i \ f \in Lip_{p(\cdot)}(G, \alpha), \quad ii) \ E_n(f)_{G,p(\cdot)} = O\left(n^{-\alpha}\right), \qquad n = 1, 2, 3, ..$$

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