Nikolskii constants for polynomials on the unit sphere

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Our main goal

- to extend recent results of Levin and Lubinsky for the (L^q, L^p) Nikolskii constant in the trigonometric case on the unit circle **to the case of the unit sphere** $\mathbb{S}^d$,
- to obtain new estimates of the spherical (L^1, L^∞) Nikolskii constant **which decay exponentially fast as** $d \to \infty$,
- to improve the asymptotic order of the spherical Nikolskii constant **for lacunary spherical polynomials**.

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The classical Nikolskii inequality

- Let $\|f\|_p := \begin{cases} \left(\int_X |f|^p\right)^{1/p}, & 0 < p < \infty, \\ \operatorname{ess\,sup}_X |f|, & p = \infty. \end{cases}$ be the norm of $f\colon X \to \mathbb{R}$.
- The Nikolskii inequality for 2π-periodical trigonometric polynomials T of degree $n \in \mathbb{N}$ reads as follows:

$$\|T\|_q \le C_{p,q} n^{\frac{1}{p}-\frac{1}{q}} \|T\|_p, \quad 0 < p < q \le \infty,$$

- A similar inequality holds for entire functions F of exponential type at most $\sigma > 0$:

$$\|F\|_q \le c_{p,q} \sigma^{\frac{1}{p}-\frac{1}{q}} \|F\|_p.$$

- For $\mathbb{T}^d$ ($\mathbb{R}^d$) it was proved by S. M. Nikolskii (1951) with constants 2^d, $p \ge 1$.

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Proof

- Let $r = \lceil p/2 \rceil$, $N = nr$, and let $K_N = (2\pi)^{-1} D_N$ be the Dirichlet kernel. Then

$$|T(x)^r| = \left| \int_{\mathbb{T}} T(\cdot)^r K_N(x - \cdot) \right| \leq \|T\|_\infty^{r-\frac{p}{2}} \int_{\mathbb{T}} |T(\cdot)|^{\frac{p}{2}} |K_N(x - \cdot)|$$

$$\overset{\text{CBS}}{\leq} \|T_n\|_\infty^{r-\frac{p}{2}} \|T\|_p^{\frac{p}{2}} \|K_N\|_2 \;\Rightarrow\; \|T\|_\infty \leq \|K_N\|_2^{\frac{2}{p}} \cdot \|T\|_p$$

$$\Leftrightarrow\; \|T\|_\infty \leq \left(\frac{2N+1}{2\pi} \right)^{\frac{1}{p}} \|T\|_p,$$

and

$$\|T\|_q \leq \|T\|_\infty^{1-\frac{p}{q}} \|T\|_p^{\frac{p}{q}} \leq \left(\frac{2nr+1}{2\pi} \right)^{\frac{1}{p}-\frac{1}{q}} \|T\|_p.$$

In particular,

$$\|T\|_q \leq \left(\frac{2n+1}{2\pi} \right)^{\frac{1}{p}-\frac{1}{q}} \|T\|_p, \quad 0 < p \leq 2. \qquad \square$$

- This approach can be applied in general situation, when $f = f * K_N$ holds with a polynomial reproducing kernel K_N.

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The case $d = 1$

- The Nikolskii inequality is sharp if $q = \infty$, $p = 2$.
- No other sharp inequalities are known, only both sides estimates. For instance, it was proved (G., 2005) that

$$n\mathcal{L}_1 \le \mathcal{C}_{1,\infty}(n) \le (n+1)\mathcal{L}_{1,\infty},$$

$$1.081 \le 2\pi\mathcal{L}_{1,\infty} \le 1.098 \quad (\text{cf. } \le 2),$$

  where

$$\mathcal{C}_{p,q}(n) := \sup\{\|T\|_q : \|T\|_p = 1\}$$

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Levin–Lubinsky result

- E. Levin and D. Lubinsky (2015) proved that

$$\lim_{n\to\infty} \frac{\mathcal{C}_{p,\infty}(n)}{n^{\frac{1}{p}}} = \mathcal{L}_{p,\infty}, \quad 0 < p < \infty,$$

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$$\liminf_{n\to\infty} \frac{\mathcal{C}_{p,\infty}(n)}{n^{\frac{1}{p}-\frac{1}{q}}} \geq \mathcal{L}_{p,q}, \quad 0 < p < q < \infty.$$

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- Recently M. Ganzburg and S. Tikhonov have generalized it for the Bernstein–Nikolskii inequality.
- The main ingredients to prove it are the Poisson summation formula

$$\sum_{k\in\mathbb{Z}} f(x+2\pi k) = \frac{1}{2\pi} \sum_{k\in\mathbb{Z}} \widehat{f}(k) e^{ikx},$$

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The case $d \geq 1$

- We extend these results to the case of the unit sphere $\mathbb{S}^d \subset \mathbb{R}^{d+1}$, $d \in \mathbb{N}$.
- Let Π_n^d be the space of all spherical polynomials of degree at most n (i.e., restrictions on $\mathbb{S}^d$ of algebraic polynomials in $d+1$ variables of total degree at most n),

$$d_n := \dim \Pi_n^d = \frac{2n^d}{\Gamma(d+1)} \left(1 + O(n^{-1})\right), \quad n \to \infty.$$

- We study the asymptotic behavior of sharp Nikolskii constant

$$\mathcal{C}_{p,q}(n,d) := \sup_{f \in \Pi_n^d,\, \|f\|_{L^p(\mathbb{S}^d)}=1} \|f\|_{L^q(\mathbb{S}^d)}$$

for $0 < p < q \leq \infty$ as $n \to \infty$.

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for $0 < p < q \leq \infty$ as $n \to \infty$.

Preliminary results

- It is well known that (A. I. Kamzolov, 1984)

$$\mathcal{C}_{p,q}(n,d) \le C_{p,q,d}\, n^{d\,(\frac{1}{p}-\frac{1}{q})}.$$

- To prove it, as above, we can use the integral representation

$$f(x) = \int_{\mathbb{S}^d} f(y) K_n(x\cdot y)\, dy,$$

where K_n is the reproducing kernel, $d_n^{-1}|\mathbb{S}^d| \cdot K_n(t) = \frac{P_n^{(\frac{d}{2},\frac{d}{2}-1)}(t)}{P_n^{(\frac{d}{2},\frac{d}{2}-1)}(1)}$ is the normalized Jacobi polynomial, $\|K_n\|_2^2 = d_n|\mathbb{S}^d|^{-1}$.
For $0 < p \le 2$, it implies

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Some interesting results

- V. V. Arestov and M. V. Deikalova (2014) studied $\mathcal{C}_{p,\infty}(n,d)$ for $p \geq 1$. They showed that the supremum here can be in fact achieved for $p \geq 1$ by zonal polynomials $P(e \cdot x)$:

$$\mathcal{C}_{p,\infty}(n,d) = \sup_{P \in \mathcal{P}_n} \frac{P(1)}{\left(|\mathbb{S}|^{d-1} \int_{-1}^{1} |P(t)|^p (1-t^2)^{\frac{d-2}{2}}\, dt\right)^{\frac{1}{p}}}.$$

An extremizer exists uniquely and it is the polynomial of least deviation from zero in $L^p([-1,1];(1-t)^{\frac{d}{2}}(1+t)^{\frac{d-2}{2}})$.

- While explicit expression for $\mathcal{C}_{p,\infty}(n,d)$ is not available (unless $p=2$), the exact value of the supremum restricted on non-negative polynomials is known for $p=1$ (see, e.g., V. I. Levenshtein, 1998):

$$|\mathbb{S}|^d\, \mathcal{C}^{+}_{1,\infty}(n,d) = \begin{cases} \binom{d+k}{d} + \binom{d+k-1}{d}, & n = 2k, \\ 2\binom{d+k}{d}, & n = 2k+1. \end{cases}$$

This constant has applications in metric geometry to obtain some tight-bounds for spherical designs (quadrature formulas).

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Theorem 1

- *For* $0 < p < \infty$ *and* $q = \infty$,

$$\lim_{n\to\infty} \frac{\mathcal{C}_{p,\infty}(n,d)}{n^{\frac{d}{p}}} = \mathcal{L}_{p,\infty}(d),$$

and for $0 < p < q < \infty$,

$$\liminf_{n\to\infty} \frac{\mathcal{C}_{p,\infty}(n,d)}{n^{d\,(\frac{1}{p}-\frac{1}{q})}} \geq \mathcal{L}_{p,q}(d),$$

where the constant $\mathcal{L}(d,p,q)$ *is defined for* $0 < p < q \leq \infty$ *by*

$$\mathcal{L}_{p,q}(d) := \sup_{f\in\mathcal{E}_p^d,\, \|f\|_{L^p(\mathbb{R}^d)}=1} \|f\|_{L^q(\mathbb{R}^d)},$$

with $\mathcal{E}_p^d$ *denoting the set of all entire functions* $f \in L^p(\mathbb{R}^d)$ *of spherical exponential type at most* 1.

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Sketch of proof

- Our proof relies on a recent deep result of A. Bondarenko, D. Radchenko and M. Viazovska (2015) on optimal asymptotic bounds for well-separated spherical designs:
  For each integer $N \ge C_d n^d$, there exists $\{x_{n,i}\}_{i=1}^N \subset \mathbb{S}^d$ such that
  $$\frac{1}{|\mathbb{S}^d|}\int_{\mathbb{S}^d} f(x)\,dx = \frac{1}{N}\sum_{i=1}^{N} f(x_{n,i}), \quad \forall f \in \Pi_n^d,$$
  and $\min_{i\neq j} \operatorname{dist}(x_{n,i}, x_{n,j}) \ge c_d N^{-1/d}$.
- Also we use $\psi(x) := \left(\frac{x}{|x|}\sin|x|, \cos|x|\right) : \pi\mathbb{B}^d \to \mathbb{S}^d$,
  $$\int_{\mathbb{S}^d} f(x)\,dx = \frac{1}{n^d}\int_{n\pi\mathbb{B}^d} f(\psi(x/n))\left(\frac{\sin|x|/n}{|x|/n}\right)^{d-1} dx,$$
  and $\frac{K_n(\cos t/n)}{K_n(1)} \underset{n\to\infty}{\rightrightarrows} j_{d/2}(t)$, where
  $j_\alpha(t) = \Gamma(\alpha+1)(\frac{t}{2})^{-\alpha} J_\alpha(t) = \prod_{k=1}^{\infty}\left(1 - \frac{t^2}{q_{\alpha,k}^2}\right)$ is the normalized Bessel function, $q_{\alpha,k}$ are positive zeros of J_α.

Sketch of proof

- Our proof relies on a recent deep result of A. Bondarenko, D. Radchenko and M. Viazovska (2015) on optimal asymptotic bounds for well-separated spherical designs:
  For each integer $N \geq C_d n^d$, there exists $\{x_{n,i}\}_{i=1}^N \subset \mathbb{S}^d$ such that

  $$\frac{1}{|\mathbb{S}^d|} \int_{\mathbb{S}^d} f(x)\,dx = \frac{1}{N} \sum_{i=1}^{N} f(x_{n,i}), \quad \forall f \in \Pi_n^d,$$

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The case $p = 1$, $q = \infty$

- We give new estimates of the normalized Nikolskii constant

$$L_d := \lim_{n\to\infty} \frac{\mathcal{C}_{1,\infty}(n,d)}{d_n|\mathbb{S}^d|^{-1}} = 2^{-1}|\mathbb{S}^d|\Gamma(d+1)\,\mathcal{L}_{1,\infty}(d).$$

- Using known estimates of $\mathcal{C}_{1,\infty}(n,d)$ (see, e.g., M. V. Deikalova, 2009) we obtain

$$e^{-d} \le L_d \le 1, \quad d \in \mathbb{N}$$

Theorem 2

We have the following sharp constant in the Nikolskii inequality for nonnegative functions

$$L_d^+ = 2^{-d}, \quad d \in \mathbb{N}.$$

To prove it we use the fact that the problem L_d^+ can be reformulated as the Turan problem for the unit ball in $\mathbb{R}^d$.

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Theorem 3

- *For any $d \in \mathbb{N}$*

$$L_d \le {}_1F_2\left(\frac{d}{2}; \frac{d}{2}+1, \frac{d}{2}+1; -\frac{\beta_d^2}{4}\right),$$

where ${}_1F_2$ is the hypergeometric function and β_d is the smallest positive zero of the Bessel function $J_{d/2}$.
In particular, this implies that the constant L_d decays exponentially fast as $d \to \infty$:

$$2^{-d} \le L_d \le (\sqrt{2/e})^{d\,(1+O(d^{-2/3}))},$$

where $\sqrt{2/e} = 0.85776\cdots$.

- For $d = 1$, we obtain T.–A.–K.–P. estimate

$$L_1 \le \frac{\operatorname{Si}\pi}{\pi} = 0.589\cdots \quad (\text{cf. } L_1 < 0.549),$$

for $d = 2$

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Sketch of proof

We use the following dual problem

$$L_d \le \inf_{\substack{1 \le r_1 < r_2 < \dots \\ a_k \in \mathbb{R}}} \Big\| j_{d/2}(\cdot) - \underbrace{\sum_{k=1}^{\infty} a_k j_{d/2-1}(r_k \,\cdot)}_{F(\cdot) \perp \mathcal{E}_1^d} \Big\|_{L^\infty(\mathbb{R}_+)}.$$

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If $r_k = \frac{q_{d/2,k}}{q_{d/1,1}}$ are normalized zeros of $J_{d/2}$, then the solution of the dual problem is ${}_1F_2\big(\frac{d}{2}; \frac{d}{2}+1, \frac{d}{2}+1; -\frac{\beta_d^2}{4}\big)$.

Graphs of the function $j_{d/2}(t) - F^*(t)$ for $d = 1$ (G., 2005), $d = 2$, and the hypothetical extremal signum-function:

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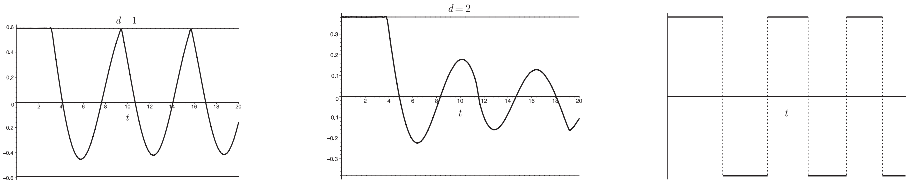

Some further results (Theorem 5)

- We observe that for $d \geq 2$, the asymptotic order of the usual Nikolskii inequality on $\mathbb{S}^d$ can be significantly improved in many cases, for lacunary spherical polynomials of the form $f = \sum_{j=0}^{m} f_{n_j}$ with f_{n_j} being a spherical harmonic of degree n_j and $n_{j+1} - n_j \geq 3$.
- For instance,

$$\|f\|_{p'} \leq C_d n^{\frac{d+1}{2}(\frac{1}{p}-\frac{1}{p'})}\|f\|_p, \quad 1 \leq p \leq 2,$$

cf. $\|f\|_{p'} \leq C_d n^{d(\frac{1}{p}-\frac{1}{p'})}\|f\|_p$.

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Thank you for your attention!