FOURIER STANDARD SPACES
A comprehensive class of function spaces

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Fourier Analysis around 1918

Fourier Series expansions have been introduced in 1822, ca. 200 years ago. Looking up was has been going on in Fourier Analysis ca. 100 years ago in Hungary one finds a paper by Friedrich Riesz:

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Publications results for "Author=(Riesz, F)"*

MR1544321 DML
Riesz, Friedrich;
Über die Fourierkoeffizienten einer stetigen Funktion von beschränkter Schw
(German)
Math. Z. 2 (1918), no. 3-4, 312–315.
Source: Springer-Verlag
Review PDF | Clipboard | Journal | Article | Make Link

He writes (in German!) that there exist continuous, periodic functions of bounded variation which do not satisfy the decay conditions $a_n = O(1/n)$ and $b_n = O(1/n)$.

Fourier Analysis around 1929

Only 11 years later Plessner was able to characterize the (classical) property of absolute continuity as equivalent for a BV-function with

$$\|F - T_x F\|_{BV} \to 0 \quad \text{for} \quad x \to 0.$$

This is what we characterize today as the property that $f = F'$ is in $\boldsymbol{L}^1$ and hence by Riemann Lebesgue we get the above condition.

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Publications results for "(Author=(Plessner)) AND pubyear=1929 "

MR1581172 DML

Plessner, A;

Eine Kennzeichnung der totalstetigen Funktionen. (German)

J. Reine Angew. Math. 160 (1929), 26–32.

Source: De Gruyter

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At the same time in Vienna

At the same time Johann Radon was publishing his famour paper on what is now called the *RADON TRANSFORM*:

`Radon, J.: Über die Bestimmung von Funktionen durch ihre Integralwerte längs gewisser Mannigfaltigkeiten.(German) JFM 46.0436.02 Leipz. Ber. 69, 262-277 (1917).`

His 1913 Habilitation thesis is entitled: Theorie und Anwendungen der absolut additiven Mengenfunktionen Sitzungsberichte der Akademie (144 p.) aims at "... creating a general theory covering on one hand the theory of linear integral equations and on the other hand linear and bilinear forms of infinitely many variables.."

At the same time Functional Analysis was born

Radon also describes measure theory as the foundation of emerging functional analysis. Poland became an important player then.

`Stefan Banach and H. Steinhaus: Sur la convergence en moyenne de series de Fourier. (Polish) JFM 47.0256.05 Krak. Anz. (A) 1918, 87-96 (1919).`

which found recognition at the international congress in 1937

`S. Banach: Die Theorie der Operationen und ihre Bedeutung für die Analysis. C. R. Congr. Int. Math. 1, 261-268 (1937)`

Norbert Wiener's book appeared 1933

By 1933 the theory of characters of Abelian groups (Paley and Wiener, and of course Pontryagin) had been established, the existence of the *Haar measure* has been introduced by Alfred Haar. In that year also Norbert Wiener's book appeared, which allowed to take a more general approach to the Fourier transform:

Wiener, Norbert: The Fourier integral and certain of its applications. (English) Zbl 0006.05401, Cambridge: Univ. Press XI, 201 S. (1933).

The relevance of L^p-spaces

If one asks, which function spaces have been used and relevant in those days the list will be quite short: Aside from BV and absolute continuity mostly the family of Lebesgue spaces appeared to be most useful for a study of the Fourier transform.
There are "good reasons". The Fourier transform is given by:

$$\hat{f}(s) := \int_{\mathbb{R}^d} f(t) e^{2\pi i s \cdot t} dt$$

appears to require $f \in \boldsymbol{L}^1(\mathbb{R}^d)$, same with convolution (integrals):

$$f * g(x) := \int_{\mathbb{R}^d} f(x-y) g(y) dy,$$

which turns $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ into a Banach algebra.

and some 50 years later ...

Hans Reiter's book on **Classical Harmonic Analysis and Locally Compact Groups** appeared in 1968, and was describing Harmonic Analysis as the

STUDY OF THE BANACH ALGEBRA $\left(\boldsymbol{L}^1(G), \|\cdot\|_1\right)$,

its behaviour under the Fourier transform, the study of closed ideals (with the hint to the problem of spectral synthesis).

Around that time (1972) Lennart Carleson was able to prove the a.e. convergence of Fourier series in $\left(\boldsymbol{L}^2(\mathbb{U}), \|\cdot\|_2\right)$.

Of course we saw the books of Katznelson, Rudin, Loomis and in particular Hewitt and Ross at the same time. Carl Herz called the comprehensive book by C. Graham and C. McGehee a "tombstone to Harmonic Analysis" (1979) (Book Review by C. Herz: Bull. Amer. Math. Soc. 7 (1982), 422425).

Where did Fourier Analysis play a role?

Not to say "everywhere in analysis" let us mention some important developments:

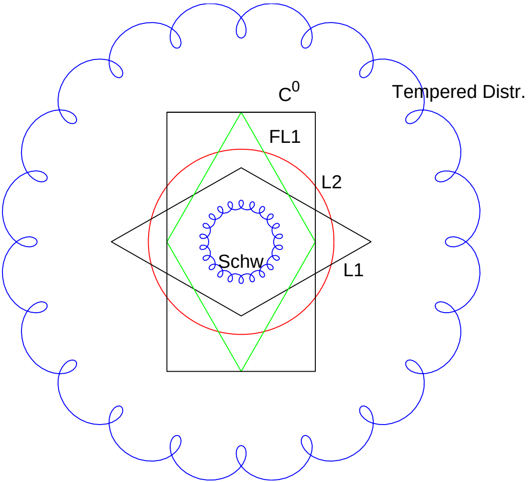
1. L. Schwartz theory of tempered distributions extended the range of the Fourier transform enormously (it was not anymore an integral transform!)
2. L. Hörmander based on this approach (influence of Marcel Riesz!) his treatment of PDEs;
3. J. Peetre an H. Triebel started the theory of function spaces, interpolation theory: Besov-Triebel-Lizorkin spaces;
4. E. Stein and his school developed the theory of maximal functions, Hardy spaces, singular integral operators;

Last 30 years ...

If one has to name a particular development related to Fourier Analysis and Function spaces one certainly has to name the new family of (orthogonal and non-orthogonal) characterizations of function spaces via atomic decompositions (resp. Banach frames).

1. it all begin with **wavelets** (1987, Yves Meyer, Abel Price 2017!), S. Mallat, and Ingrid Daubechies;
2. Gabor Analysis (D. Gabor: 1946, mathematics since ca. 1980!, A.J.E.M.Janssen, members of NuHAG/Vienna);
3. Shearlets, curvlets, Blaschke group, *coorbit theory*...
4. Felix Voigtländer (PhD 2016, Aachen): decomposition spaces and abstract wavelet spaces.

The Schwartz Gelfand triple



New problems require new function spaces

The discussion about Gabor analysis did require a new family of function spaces (which had fortunately been developed already since 1980 in the work of the speaker, starting with the theory of **Wiener Amalgam spaces** which had been the crucial step towards a general theory of **modulation spaces**, with the classical family $\left(\boldsymbol{M}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^s_{p,q}}\right), 1 \leq p, q \leq \infty$ showing a lot of similarity to the family of *Besov spaces* $(\boldsymbol{B}^s_{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{B}^s_{p,q}})$.

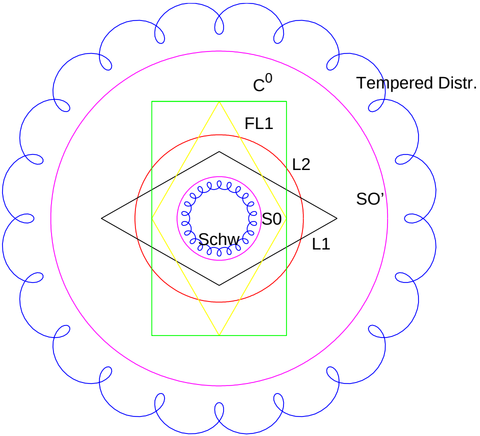
Nowadays it is clear that these spaces are not only well suited for the description of *pseudo-differential* and *Fourier integral operators* or the Schrödinger equation, but one starts to look at their usefulness in the context of more classical settings, but also with respect to the design of (mathematically well founded) courses for engineers!

The universe of Fourier Standard Spaces

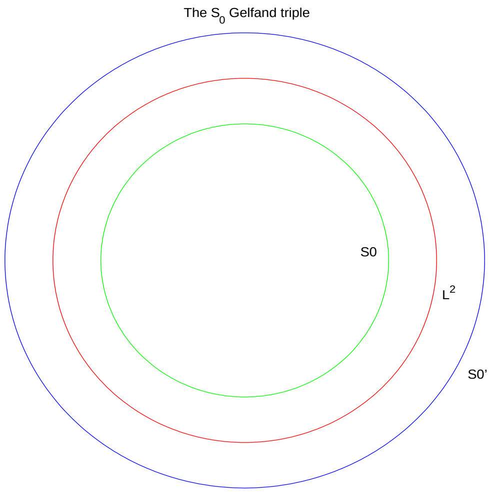
As opposed to the one-paramter family of $\boldsymbol{L}^p$-spaces over $\mathbb{R}^d$ which do not show inclusions in any direction (sometimes for *local* reasons, sometimes for *global* reasons, the Wiener amalgam spaces $\boldsymbol{W}(\boldsymbol{L}^p, \ell^q)(\mathbb{R}^d)$ but also the modulation spaces $(\boldsymbol{M}^{p,q}(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{M}^{p,q}})$ show natural (and proper) inclusions.
We will concentrate on the unweighted case (i.e. $s = 0$), and there the minimal space is the Segal algebra $\boldsymbol{S}_0(\mathbb{R}^d) = \boldsymbol{M}_0^{1,1}(\mathbb{R}^d) = \boldsymbol{M}^1(\mathbb{R}^d)$ and the maximal space is its dual, the space $\boldsymbol{S}_0'(\mathbb{R}^d) = \boldsymbol{M}^{\infty,\infty}(\mathbb{R}^d) = \boldsymbol{M}^\infty(\mathbb{R}^d)$.

We plan to look closer at the space in between these two spaces! First let us introduce the concept of a Banach Gelfand triple and see how these spaces compare to established spaces!

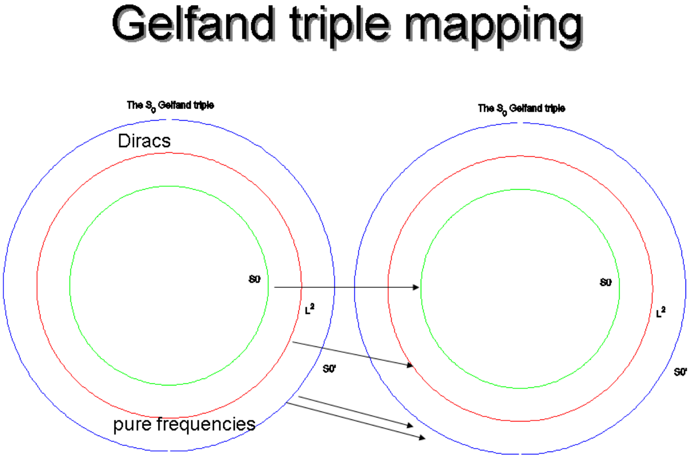
An enriched schematic description



The Banach Gelfand Triple $(S_0, L^2, S_0')(\mathbb{R}^d)$



A pictorial presentation of the BGTr morphism



BANACH GELFAND TRIPLES: a new category

Definition

A triple, consisting of a Banach space $\boldsymbol{B}$, which is dense in some Hilbert space $\mathcal{H}$, which in turn is contained in $\boldsymbol{B}'$ is called a Banach Gelfand triple.

Definition

If $(\boldsymbol{B}_1, \mathcal{H}_1, \boldsymbol{B}_1')$ and $(\boldsymbol{B}_2, \mathcal{H}_2, \boldsymbol{B}_2')$ are Gelfand triples then a linear operator T is called a [unitary] Gelfand triple isomorphism if

1. A is an isomorphism between $\boldsymbol{B}_1$ and $\boldsymbol{B}_2$.
2. A is [a unitary operator resp.] an isomorphism between $\mathcal{H}_1$ and $\mathcal{H}_2$.
3. A extends to norm-to-norm continuous isomorphism between $\boldsymbol{B}_1'$ and $\boldsymbol{B}_2'$ which is then IN ADDITION w^*-w^*--continuous!

Banach Gelfand Triples, the prototype

In principle every CONB (= *complete orthonormal basis*) $\Psi = (\psi_i)_{i \in I}$ for a given Hilbert space $\mathcal{H}$ can be used to establish such a unitary isomorphism, by choosing as $\boldsymbol{B}$ the space of elements within $\mathcal{H}$ which have an absolutely convergent expansion, i.e. satisfy $\sum_{i \in I} |\langle x, \psi_i \rangle| < \infty$.
For the case of the Fourier system as CONB for $\mathcal{H} = \boldsymbol{L}^2([0,1])$, i.e. the corresponding definition is already around since the times of N. Wiener: $\boldsymbol{A}(\mathbb{U})$, the space of absolutely continuous Fourier series. It is also not surprising in retrospect to see that the dual space $\boldsymbol{PM}(\mathbb{U}) = \boldsymbol{A}(\mathbb{U})'$ is space of *pseudo-measures*. One can extend the classical Fourier transform to this space, and in fact interpret this extended mapping, in conjunction with the classical Plancherel theorem as the first unitary Banach Gelfand triple isomorphism, between $(\boldsymbol{A}, \boldsymbol{L}^2, \boldsymbol{PM})(\mathbb{U})$ and $(\ell^1, \ell^2, \ell^\infty)(\mathbb{Z})$.

The Segal Algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$, 1979

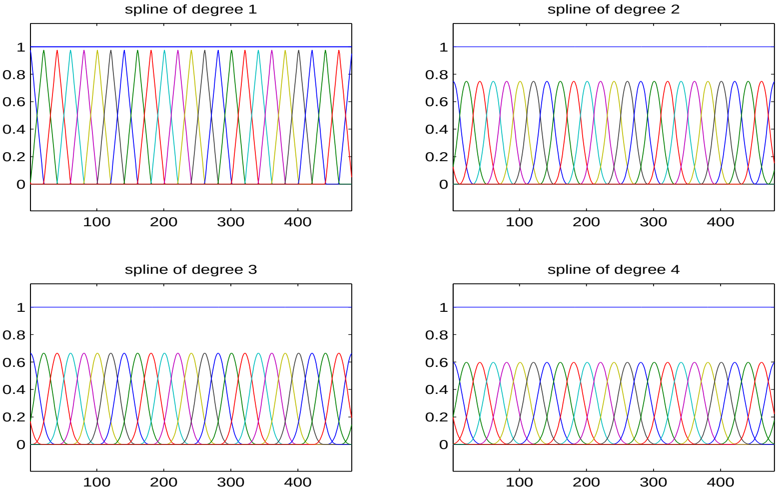
In the last 2-3 decades the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$ (equal to the modulation space $(M^1(\mathbb{R}^d), \|\cdot\|_{M^1})$) and its dual, $(S_0'(\mathbb{R}^d), \|\cdot\|_{S_0'})$ have gained importance for many questions of Gabor analysis or time-frequency analysis in general.

It can be characterized as the smallest (non-trivial) Banach space of (continuous and integrable) functions with the property, that time-frequency shifts acts isometrically on its elements, i.e. with

$$\|T_x f\|_B = \|f\|_B, \quad \text{and} \quad \|M_s f\|_B = \|f\|_B, \quad \forall f \in B,$$

where T_x is the usual translation operator, and M_s is the *frequency shift* operator, i.e. $M_s f(t) = e^{2\pi i s \cdot t} f(t), t \in \mathbb{R}^d$. This description implies that $S_0(\mathbb{R}^d)$ is also Fourier invariant!

Illustration of the B-splines providing BUPUs



The Segal Algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$: description

There are many different ways to describe $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$. Originally it has been introduced as *Wiener amalgam space* $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$, but the standard approach is to describe it via the STFT (short-time Fourier transform) using a Gaussian window given by $g_0(t) = e^{-\pi|t|^2}$.

A short description of the Wiener Amalgam space for $d = 1$ is as follows: Starting from the basis of B-splines of order ≥ 2 (e.g. triangular functions or cubic B-splines), which form a (smooth and uniform) partition of the form $(\varphi_n) := (T_n\varphi)_{n\in\mathbb{Z}}$ we can say that $f \in \mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d)$ belongs to $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)(\mathbb{R}^d)$ if and only if

$$\|f\| := \sum_{n\in\mathbb{Z}} \|\widehat{f \cdot \varphi_n}\|_{\boldsymbol{L}^1} < \infty.$$

Using tensor products the definition extends to $d \geq 2$.

Banach Gelfand Triple (S_0, L^2, S_0'): BASICS

Let us collect a few facts concerning this Banach Gelfand Triple (BGTr), based on the Segal algebra $\left(S_0(\mathbb{R}^d), \|\cdot\|_{S_0}\right)$:

- $S_0(\mathbb{R}^d)$ is dense in $\left(L^2(\mathbb{R}^d), \|\cdot\|_2\right)$, in fact within any $\left(L^p(\mathbb{R}^d), \|\cdot\|_p\right)$, with $1 \leq p < \infty$ (or in $\left(C_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$);
- Any of the L^p-spaces, with $1 \leq p \leq \infty$ is continuously embedded into $S_0'(\mathbb{R}^d)$;
- Any translation bounded measure belongs to $S_0'(\mathbb{R}^d)$, in particular any Dirac-comb $⊔⊔_\Lambda := \sum_{\lambda\in\Lambda} \delta_\lambda$, for $\Lambda \lhd \mathbb{R}^d$.
- $S_0(\mathbb{R}^d)$ is w^*-dense in $S_0'(\mathbb{R}^d)$, i.e. for any $\sigma \in S_0'(\mathbb{R}^d)$ there exists a sequence of test functions s_n in $S_0(\mathbb{R}^d)$ such that

$$\int_{\mathbb{R}^d} f(x)s_n(x)dx \to \sigma(f), \quad \forall f \in S_0(\mathbb{R}^d). \tag{1}$$

The key-players for time-frequency analysis

Time-shifts and Frequency shifts

$$T_x f(t) = f(t-x)$$

and $x, \omega, t \in \mathbb{R}^d$

$$M_\omega f(t) = e^{2\pi i \omega \cdot t} f(t) .$$

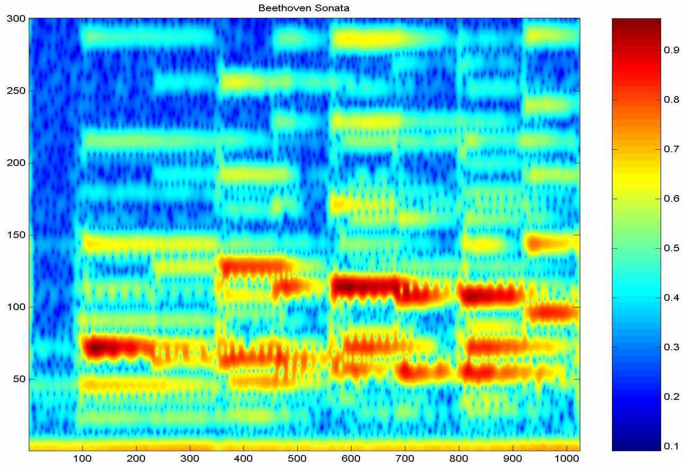
Behavior under Fourier transform

$$(T_x f)\hat{} = M_{-x} \hat{f} \qquad (M_\omega f)\hat{} = T_\omega \hat{f}$$

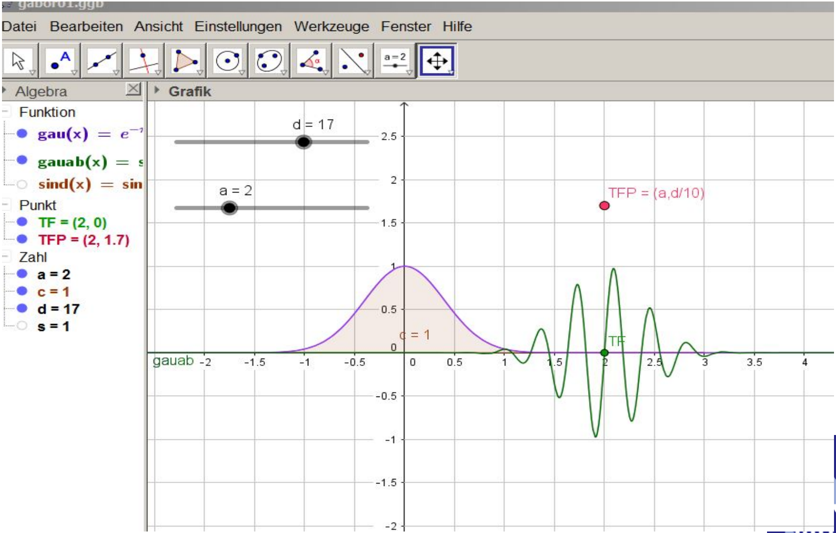
The Short-Time Fourier Transform

$$V_g f(\lambda) = \langle f, M_\omega T_t g \rangle = \langle f, \pi(\lambda) g \rangle = \langle f, g_\lambda \rangle, \;\; \lambda = (t, \omega);$$

A Typical Musical STFT



Demonstration using GEOGEBRA (very easy to use!!)



Spectrogramm versus Gabor Analysis

Assuming that we use as a "window" a Schwartz function $g \in \mathcal{S}(\mathbb{R}^d)$, or even the Gauss function $g_0(t) = exp(-\pi|t|^2)$, we can define the spectrogram for general tempered distributions $f \in \mathcal{S}'(\mathbb{R}^d)$! It is a continuous function over *phase space*.

In fact, for the case of the Gauss function it is *analytic* and in fact a member of the *Fock space*, of interest within complex analysis.

Both from a pratical point of view and in view of this good smoothness one may expect that it is enough to *sample this spectrogram*, denoted by $V_g(f)$ and still be able to reconstruct f (in analogy to the reconstruction of a band-limited signal from regular samples, according to Shannon's theorem).

So let us start from the continuous spectrogram

The spectrogram $V_g(f)$, with $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$ is well defined and has a number of good properties. Cauchy-Schwarz implies:

$$\|V_g(f)\|_\infty \leq \|f\|_2 \|g\|_2, \quad f, g \in \boldsymbol{L}^2(\mathbb{R}^d),$$

in fact $V_g(f) \in \boldsymbol{C}_0(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$. We have the **Moyal identity**

$$\|V_g(f)\|_2 = \|g\|_2 \|f\|_2, \quad g, f \in \boldsymbol{L}^2(\mathbb{R}^d).$$

Since assuming that g is normalized in $\boldsymbol{L}^2(\mathbb{R}^d)$, or $\|g\|_2$ is no problem we will assume this from now on.
Note: $V_g(f)$ is a complex-valued function, so we usually look at $|V_g(f)|$, or perhaps better $|V_g(f)|^2$, which can be viewed as a probability distribution over $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ if $\|f\|_2 = 1 = \|g\|_2$.

The continuous reconstruction formula

Now we can apply a simple abstract principle: Given an isometric embedding T of $\mathcal{H}_1$ into $\mathcal{H}_2$ the inverse (in the range) is given by the adjoint operator $T^* : \mathcal{H}_2 \to \mathcal{H}_1$, simply because

$$\langle h, h \rangle_{\mathcal{H}_1} = \|h\|^2_{\mathcal{H}_1} = (!) \, \|Th\|^2_{\mathcal{H}_2} = \langle Th, Th \rangle_{\mathcal{H}_2} = \langle h, T^*Th \rangle_{\mathcal{H}_1}, \; \forall h \in \mathcal{H}_1,$$

and thus by the *polarization principle* $T^*T = Id$
In our setting we have (assuming $\|g\|_2 = 1$) $\mathcal{H}_1 = \boldsymbol{L}^2(\mathbb{R}^d)$ and $\mathcal{H}_2 = \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, and $T = V_g$. It is easy to check that

$$V_g^*(F) = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(\lambda)\pi(\lambda)g \, d\lambda, \quad F \in \boldsymbol{L}^2(\mathbb{R}^d \times \widehat{\mathbb{R}}^d), \qquad (2)$$

understood in the weak sense, i.e. for $h \in \boldsymbol{L}^2(\mathbb{R}^d)$ we expect:

$$\langle V_g^*(F), h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} F(x) \cdot \langle \pi(\lambda)g, h \rangle_{\boldsymbol{L}^2(\mathbb{R}^d)} d\lambda. \qquad (3)$$

Continuous reconstruction formula II

Putting things together we have

$$\langle f, h\rangle = \langle V_g^*(V_g(f)), h\rangle = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} V_g(f)(\lambda) \cdot \overline{V_g(h)(\lambda)}\, d\lambda. \qquad (4)$$

A more suggestive presentation uses the symbol $g_\lambda := \pi(\lambda)g$ and describes the inversion formula for $\|g\|_2 = 1$ as:

$$f = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \langle f, g_\lambda\rangle\, g_\lambda\, d\lambda, \quad f \in \boldsymbol{L}^2(\mathbb{R}^d). \qquad (5)$$

This is quite analogous to the situation of the Fourier transform

$$(6) \quad f = \int_{\mathbb{R}^d} \langle f, \chi_s\rangle\, \chi_s\, ds, = \int_{\mathbb{R}^d} \hat{f}(x) e^{2\pi i s \cdot} ds \quad f \in \boldsymbol{L}^2(\mathbb{R}^d), \quad (6)$$

with $\chi_s(t) = exp(2\pi i \langle s, t\rangle)$, $t, s \in \mathbb{R}^d$, describing the "pure frequencies" (plane waves, resp. *characters* of $\mathbb{R}^d$).

Characterizing $S_0(\mathbb{R}^d)$ via the STFT

While $V_g(f)$ is a continuous and square integrable function for any pair $g, f \in \boldsymbol{L}^2(\mathbb{R}^d)$, the choice of $g = g_0$, with $g_0(t) = e^{-\pi|t|^2}$, the usual Gaussian function is particularly pleasing, in fact in this case $V_{g_0}(f)$ is even an analytic function ($>>$ Fock spaces).
For certain $\boldsymbol{L}^1(\mathbb{R})$-functions, like step functions f, their STFT $V_{g_0}(f)$ is *not integrable*, same for the SINC-function (bad decay at infinity), *but if* $V_{g_0}(f) \in \boldsymbol{L}^1(\mathbb{R}^{2d})$ it means that the STFT is reasonably well concentrated within phase space.

Theorem

$f \in \boldsymbol{L}^2(\mathbb{R}^d)$ belongs to $\boldsymbol{S}_0(\mathbb{R}^d)$ if and only if $V_{g_0}(f) \in \boldsymbol{L}^1(\mathbb{R}^{2d})$. Moreover $\|V_{g_0} f\|_{\boldsymbol{L}^1}$ is an equivalent norm on $\boldsymbol{S}_0(\mathbb{R}^d)$. $\mathcal{S}(\mathbb{R}^d)$ is a dense subspace of the Banach space $\left(\boldsymbol{S}_0(\mathbb{R}^d), \|\cdot\|_{\boldsymbol{S}_0}\right)$.

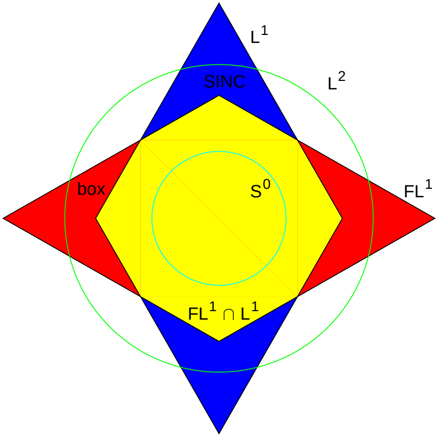
Characterization of $S_0'(\mathbb{R}^d)$ and w^*-convergence

A tempered distribution $\sigma \in \mathcal{S}'(\mathbb{R}^d)$ belongs to $S_0'(\mathbb{R}^d)$ if and only if its (continuous) STFT is a *bounded* function. Furthermore convergence corresponds to uniform convergence of the spectrogram (different windows give equivalent norms!).
We can also extend the Fourier transform form $S_0(\mathbb{R}^d)$ to $S_0'(\mathbb{R}^d)$ via the usual formula $\hat{\sigma}(f) := \sigma(\hat{f})$.
The weaker convergence, arising from the functional analytic concept of w^*-convergence has the following very natural characterization: A (bounded) sequence σ_n is w^*- convergence to σ_0 if and only if for one (resp. every) $S_0(\mathbb{R}^d)$-window g one has

$$V_g(\sigma_n)(\lambda) \to V_g(\sigma_0)(\lambda) \quad \text{for} n \to \infty,$$

uniformly over compact subsets of phase-space.

$L^1(\mathbb{R}^d)$ and the Fourier Algebra $\mathcal{F}L^1(\mathbb{R}^d)$



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Fourier Standard Spaces, the Idea

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, continuously embedded between $\boldsymbol{S}_0(G)$ and $(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'})$, i.e. with

$$\left(\boldsymbol{S}_0(G), \|\cdot\|_{\boldsymbol{S}_0}\right) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \left(\boldsymbol{S}_0'(G), \|\cdot\|_{\boldsymbol{S}_0'}\right) \tag{7}$$

is called a **Fourier Standard Space** on G (FSS of FoSS) if it has a double module structure over $(\boldsymbol{M}_b(G), \|\cdot\|_{\boldsymbol{M}_b})$ with respect to convolution and over the (Fourier-Stieltjes algebra) $\mathcal{F}(\boldsymbol{M}_b(\widehat{G}))$ with respect to pointwise multiplication.

Typically we just require that in addition to (7) one has:

$$\boldsymbol{L}^1 * \boldsymbol{B} \subseteq \boldsymbol{B} \quad \text{and} \quad \mathcal{F}\boldsymbol{L}^1 \cdot \boldsymbol{B} \subseteq \boldsymbol{B}. \tag{8}$$

Constructions within the FSS Family

1. Taking **Fourier transforms**;
2. Conditional dual spaces, i.e. the **dual space** of the closure of $\boldsymbol{S}_0(G)$ within $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$;
3. With two spaces $\boldsymbol{B}^1, \boldsymbol{B}^2$: take **intersection or sum**
4. forming **amalgam spaces** $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$; e.g. $\boldsymbol{W}(\mathcal{F}\boldsymbol{L}^1, \ell^1)$;
5. defining pointwise or convolution **multipliers**;
6. using complex (or real) **interpolation methods**, so that we get the spaces $\boldsymbol{M}^{p,p} = \boldsymbol{W}(\mathcal{F}\boldsymbol{L}^p, \ell^p)$ (all Fourier invariant);
7. any **metaplectic** image of such a space, e.g. the **fractional Fourier transform**.

Recalling the Wiener Amalgam Concept

We recall the concept of BUPUs, ideally as translates along a lattice ($T_|lambda\varphi$), with compact support and a certain amount of smoothness, perhaps cubic B-splines.
The Wiener amalgam space $\boldsymbol{W}(\boldsymbol{B},\ell^q)$ is defined as the set

$$\{f \in \boldsymbol{B}_{loc} \,|\, \|f|\boldsymbol{W}(\boldsymbol{B},\ell^q)\| := (\sum_{\lambda\in\Lambda} \|f \cdot T_\lambda\varphi\|_{\boldsymbol{B}}^q)^{1/q}\}$$

There are many "natural results" concerning Wiener amalgam spaces ([3]), namely coordinatewise action, e.g.

- duality (if test functions are dense and $q < \infty$;
- convolution and pointwise multiplication;
- interpolation (real or complex).

FOURIER STANDARD SPACES: II

The spaces in this family are useful for a discussion of questions in Gabor Analysis, which is an important branch of time-frequency analysis, but also for problems of classical Fourier Analysis, such as the discussion of Fourier multipliers, Fourier inversion questions and so on. Thus among others the space $\mathbf{L}^1(\mathbb{R}^d) \cap \mathcal{F}\mathbf{L}^1(\mathbb{R}^d)$.

Within the family there are two subfamilies, namely the **Wiener amalgam spaces** and the so-called **modulation spaces**, among them the Segal algebra $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$ or **Wiener's algebra** $\left(\mathbf{W}(\mathbf{C}_0, \ell^1)(\mathbb{R}^d), \|\cdot\|_{\mathbf{W}}\right)$.

TF-homogeneous Banach Spaces

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with

$$\mathcal{S}(\mathbb{R}^d) \hookrightarrow (\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}) \hookrightarrow \mathcal{S}'(\mathbb{R}^d)$$

is called a **TF-homogeneous Banach space** if $\mathcal{S}(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ and TF-shifts act isometrically on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, i.e. if

$$\|\pi(\lambda)f\|_{\boldsymbol{B}} = \|f\|_{\boldsymbol{B}}, \quad \forall \lambda \in \mathbb{R}^d \times \widehat{\mathbb{R}}^d, \ f \in \boldsymbol{B}. \tag{9}$$

For such spaces the mapping $\lambda \to \pi(\lambda)f$ is continuous from $\mathbb{R}^d \times \widehat{\mathbb{R}}^d$ to $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$. If it is not continuous one often has the *adjoint action* on the dual space of such TF-homogeneous Banach spaces (e.g. $\left(\boldsymbol{L}^\infty(\mathbb{R}^d), \|\cdot\|_\infty\right)$).

TF-homogeneous Banach Spaces II

An important fact concerning this family is the minimality property of the Segal algebra $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right)$.

Theorem

There is a smallest member in the family of all TF-homogeneous Banach spaces, namely the Segal algebra $\left(\mathbf{S}_0(\mathbb{R}^d), \|\cdot\|_{\mathbf{S}_0}\right) = \mathbf{W}(\mathcal{F}\mathbf{L}^1, \ell^1)(\mathbb{R}^d)$.

Justifying the properties of the family

There is a large number of results concerning $\boldsymbol{S}_0(\mathbb{R}^d)$ (defined for any dimension, but in fact for any LCA group):

$$\mathcal{F}_G \boldsymbol{S}_0(G) = \boldsymbol{S}_0(\widehat{G}).$$

There is a tensor product property, namely

$$\boldsymbol{S}_0(\mathbb{R}^{2d}) = \boldsymbol{S}_0(\mathbb{R}^d) \widehat{\otimes} \boldsymbol{S}_0(\mathbb{R}^d).$$

Multipliers are easily characterized:

Theorem

The continuous linear operators from $\boldsymbol{S}_0(\mathbb{R}^d)$ to $\boldsymbol{S}_0'(\mathbb{R}^d)$ are exactly the convolution operators by "kernels" $\sigma \in \boldsymbol{S}_0'(\mathbb{R}^d)$, with equivalence of norms (the operator norm of the convolution operator, resp. translation invariant linear system) and the $\boldsymbol{S}_0'$-norm of the corresponding convolution kernel $\|\sigma\|_{\boldsymbol{S}_0'}$.

The Kernel Theorem

It is clear that such operators between functions on $\mathbb{R}^d$ cannot all be represented by integral kernels using locally integrable $K(x, y)$ in the form

$$Tf(x) = \int_{\mathbb{R}^d} K(x, y) f(y) dy, \quad x, y \in \mathbb{R}^d, \tag{10}$$

because clearly multiplication operators should have their support on the main diagonal, but $\{(x, x) \,|\, x \in \mathbb{R}^d\}$ is just a set of measure zero in $\mathbb{R}^d \times \mathbb{R}^d$!

Also the expected "rule" to find the kernel, namely

$$K(x, y) = T(\delta_y)(x) = \delta_x(T(\delta_y)) \tag{11}$$

might not be meaningful at all.

The Hilbert Schmidt Version

There are two ways out of this problem

- restrict the class of operators
- enlarge the class of possible kernels

The first one is a classical result, i.e. the characterization of the class $\mathcal{HS}$ of Hilbert Schmidt operators.

Theorem

A linear operator T on $\left(\boldsymbol{L}^2(\mathbb{R}^d), \|\cdot\|_2\right)$ is a Hilbert-Schmidt operator, i.e. is a compact operator with the sequence of singular values in ℓ^2 if and only if it is an integral operator of the form (10) with $K \in \boldsymbol{L}^2(\mathbb{R}^d \times \mathbb{R}^d)$. In fact, we have a unitary mapping $T \to K(x,y)$, where $\mathcal{HS}$ is endowed with the Hilbert-Schmidt scalar product $\langle T, S\rangle_{\mathcal{HS}} := \mathrm{trace}(T \circ S^*)$.

The Schwartz Kernel Theorem

The other well known version of the kernel theorem makes use of the *nuclearity* of the *Frechet space* $\mathcal{S}(\mathbb{R}^d)$ (so to say the complicated topological properties of the system of seminorms defining the topology on $\mathcal{S}(\mathbb{R}^d)$).
Note that the description cannot be given anymore in the form (10) but has to replaced by a "weak description". This is part of the following well-known result due to L. Schwartz.

Theorem

There is a natural isomorphism between the vector space of all linear operators from $\mathcal{S}(\mathbb{R}^d)$ into $\mathcal{S}'(\mathbb{R}^d)$, i.e. $\mathcal{L}(\mathcal{S}, \mathcal{S}')$, and the elements of $\mathcal{S}'(\mathbb{R}^{2d})$, via $\langle Tf, g\rangle = \langle K, f \otimes g\rangle$, for $f, g \in \mathcal{S}(\mathbb{R}^d)$.

The S_0-KERNEL THEOREM

In the current setting we can describe the kernel theorem as a unitary Banach Gelfand Triple isomorphism, between operator and their (distributional) kernels, extending the classical Hilbert Schmidt version.

First we observe that $\boldsymbol{S}_0$-kernels can be identified with $\mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0')$, i.e. the *regularizing operators* from $\boldsymbol{S}_0'(\mathbb{R}^d)$ to $\boldsymbol{S}_0(\mathbb{R}^d)$, even mapping bounded and w^*- convergent nets into norm convergent sets. For those kernels also the recovery formula (11) is valid.

Theorem

The unitary Hilbert-Schmidt kernel isomorphisms extends in a unique way to a Banach Gelfand Triple isomorphism between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ *and* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \mathbb{R}^d)$.

The Spreading Representation

The **spreading representation** of operators interpretation. In some sense it can be viewed as a kind of Fourier Transform for operators. For the case of $G = \mathbb{Z}_N$ we have N^2 time-frequency shift operators (cyclic shifts combined with pointwise multiplication by pure frequencies), and in fact they form an orthonormal basis for the (Euclidean) space of $N \times N$-matrices (linear operators on $\mathbb{C}^N$), with the Frobenius scalar product.

Theorem

There is a unique (unitary) Banach Gelfand triple isomorphism between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ *and* $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, *which maps the time frequency shift operators* $\pi(\lambda) := M_\omega T_t$ *to the Dirac measures* $\delta_{t,\omega} \in \boldsymbol{S}_0'(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$.

Spreading Representation II

This also tells us, that an operator $T \in \mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0)$ is regularizing if it can be written as an operator-valued Riemannian integral

$$T = \int_{\mathbb{R}^d \times \widehat{\mathbb{R}}^d} \eta(\lambda)\pi(\lambda) d\lambda. \tag{12}$$

Of course one can also write explicit formulas (involving various transformations and partial Fourier transform) for the transition between the kernel of an operator T and its spreading "function" $\eta(T)$ (cf. [5]) which are valid in the pointwise sense (using standard integration theory), while one has to extend it to the Hilbert space case by continuity (like the usual proofs of Plancherel's theorem) and then extend it to the *outer layer* via duality (or w^*-continuity). See also [2]

The Kohn-Nirenberg Symbol

For various applications in the area of *pseudo-differntial operators* and for applications in Gabor Analysis also the so-called **Kohn-Nirenberg Symbol** $\sigma(T)$ of an operator T is of interest. It is obtained from the spreading representation via the so-called *symplectic Fourier transform*.

Theorem

The (unitary)KNS Banach Gelfand triple isomorphism between $(\mathcal{L}(\boldsymbol{S}_0', \boldsymbol{S}_0), \mathcal{HS}, \mathcal{L}(\boldsymbol{S}_0, \boldsymbol{S}_0'))$ and $(\boldsymbol{S}_0, \boldsymbol{L}^2, \boldsymbol{S}_0')(\mathbb{R}^d \times \widehat{\mathbb{R}}^d)$, $T \to \sigma(T)$ has the following covariance property:

$$\sigma[\pi(\lambda) \circ T \circ \pi(\lambda)'] = T_{t,\omega}\sigma(T).$$

Fourier Standard spaces on $\mathbb{R}^{2d}$

The kernel theorem provides a variety of new possibilities to define Fourier Standard spaces on product domains, e.g. (for simplicity) on $\mathbb{R}^d \times \mathbb{R}^d$. Among others we have

1. Given two Fourier standard spaces on $\mathbb{R}^d$ the kernels (or the Kohn-Nirenberg symbols, etc.) of the space of all linear operators $\boldsymbol{L}(\boldsymbol{B}^1, \boldsymbol{B}^2)$ is a FouSS;
2. Given any **operator ideal** within $\mathcal{L}(\mathcal{H})$ defines a corresponding FouSS of kernels (e.g. Schatten S_p-classes, various types of nuclear operators etc., see book of Pietsch);
3. some of them can be periodized (or restricted) along certain subgroups, e.g. die diagonal $\{(x,x) \,|\, , x \in \mathbb{R}\}$. The resulting spaces are then comparable with the **Herz algebras** $\boldsymbol{A}_p(\mathbb{R}^d)$, characterizing the pre-dual of all $\boldsymbol{L}^p$-multipliers.

What kind of questions can we ask?

There are at least two major type of questions which one can ask, related to the possibility of creating new spaces within the family. The key constructions have to do with

1. Wiener amalgam spaces of the form $\boldsymbol{W}(\boldsymbol{B}, \ell^q)$;
2. the double module structure on these spaces.

Let us recall that for the construction of Wiener amalgam spaces we only need the possibility of applying a BUPU (a bounded partition of unity), in our case boundedness refers to $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$.
Moreover there is a clear chain of proper inclusions then, with

$$\boldsymbol{W}(\boldsymbol{B}, \ell^1) \subsetneq \boldsymbol{W}(\boldsymbol{B}, \ell^{p_1})$$

Lower and Upper Index of a Function Space

We define (compare [4]) the lower and upper index of such a space is called

Definition

$$\mathrm{low}(\boldsymbol{B}) := \sup\{r \mid \boldsymbol{B} \subseteq W(\boldsymbol{B}, \ell^r)\}.$$

Definition

The *upper index* of $\boldsymbol{B}$ is defined as follows:

$$\mathrm{upp}(\boldsymbol{B}) := \inf\{s \mid W(\boldsymbol{B}, \ell^s) \subseteq \boldsymbol{B}\}.$$

Indices of concrete function spaces

Lemma

- *Clearly* $\mathrm{low}(\boldsymbol{L}^p) = p = \mathrm{upp}(\boldsymbol{L}^p)$, *for* $1 \leq p \leq \infty$; *in fact* $\boldsymbol{W}(\boldsymbol{L}^p, \ell^p) = \boldsymbol{L}^p(\mathbb{R}^d)$ *with norm equivalence;*
- *For* $1 \leq p \leq 2$ *one has*

$$\mathrm{low}(\mathcal{F}\boldsymbol{L}^p) = p, \quad \mathrm{upp}(\mathcal{F}\boldsymbol{L}^p) = p'.$$

For the space of multipliers we only know for sure: Since

$$\boldsymbol{B}^1 = \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{L}^1(\mathbb{R}^d)) = \boldsymbol{M}_b(\mathbb{R}^d) = \boldsymbol{W}(\boldsymbol{M}_b(\mathbb{R}^d), \ell^1)$$

$$\mathrm{low}(\boldsymbol{B}^1) = 1 = \mathrm{upp}(\boldsymbol{B}^1)$$

while in contrast for $p = 2$ we have
$\boldsymbol{B}^2 = \mathcal{H}_{\mathbb{R}^d}(\boldsymbol{L}^2(\mathbb{R}^d)) = \mathcal{F}\boldsymbol{L}^\infty(\mathbb{R}^d)$ and hence

$$\mathrm{low}(\boldsymbol{B}^2) = 1, \quad \mathrm{upp}(\boldsymbol{B}^2) = \infty.$$

Banach Module Terminology

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach module* over a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if one has a bilinear mapping $(a, b) \mapsto a \bullet b$, from $\boldsymbol{A} \times \boldsymbol{B}$ into $\boldsymbol{B}$ bilinear and associative, such that

$$\|a \bullet b\|_{\boldsymbol{B}} \leq \|a\|_{\boldsymbol{A}} \|b\|_{\boldsymbol{B}} \quad \forall\, a \in \boldsymbol{A}, b \in \boldsymbol{B}, \tag{13}$$

$$a_1 \bullet (a_2 \bullet b) = (a_1 \cdot a_2) \bullet b \quad \forall a_1, a_2 \in \boldsymbol{A}, b \in \boldsymbol{B}. \tag{14}$$

Definition

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is a *Banach ideal* in (or within, or of) a Banach algebra $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$ if $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is continuously embedded into $(\boldsymbol{A}, \cdot, \|\cdot\|_{\boldsymbol{A}})$, and if in addition (13) is valid with respect to the internal multiplication inherited from $\boldsymbol{A}$.

Wendel's Theorem

Theorem

The space of $\mathcal{H}_{\mathbf{L}^1}(\mathbf{L}^1, \mathbf{L}^1)$ *all bounded linear operators on* $\mathbf{L}^1(G)$ *which commute with translations (or equivalently: with convolutions) is naturally and isometrically identified with* $(\mathbf{M}_b(G), \|\cdot\|_{\mathbf{M}_b})$*. In terms of our formulas this means*

$$\mathcal{H}_{\mathbf{L}^1}(\mathbf{L}^1, \mathbf{L}^1)(\mathbb{R}^d) \simeq (\mathbf{M}_b(\mathbb{R}^d), \|\cdot\|_{\mathbf{M}_b}),$$

$$\textit{via } T \simeq C_\mu : f \mapsto \mu * f, \quad f \in \mathbf{L}^1, \mu \in \mathbf{M}_b(\mathbb{R}^d).$$

Lemma

$$B_{\mathbf{L}^1} = \{f \in \mathbf{B} \mid \|T_x f - f\|_{\mathbf{B}} \to 0, \ forx \to 0\}.$$

Consequently we have $(\mathbf{M}_b(\mathbb{R}^d))_{\mathbf{L}^1} = \mathbf{L}^1(\mathbb{R}^d)$*, the closed ideal of absolutely continuous bounded measures on* $\mathbb{R}^d$*.*

Pointwise Multipliers

Via the Fourier transform we have similar statements for the Fourier algebra, involving the *Fourier Stieltjes algebra*.

$$\mathcal{H}_{\mathcal{F}\boldsymbol{L}^1}(\mathcal{F}\boldsymbol{L}^1, \mathcal{F}\boldsymbol{L}^1) = \mathcal{F}(\boldsymbol{M}_b(\mathbb{R}^d)), \quad \mathcal{F}(\boldsymbol{M}_b(\mathbb{R}^d))_{\mathcal{F}\boldsymbol{L}^1} = \mathcal{F}\boldsymbol{L}^1. \quad (15)$$

Theorem

The completion of $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ *(viewed as a Banach algebra and module over itself) is given by*

$$\mathcal{H}_{\boldsymbol{C}_0}(\boldsymbol{C}_0, \boldsymbol{C}_0) = \left(\boldsymbol{C}_b(\mathbb{R}^d), \|\cdot\|_\infty\right).$$

On the other hand we have $(\boldsymbol{C}_b(\mathbb{R}^d))_{\boldsymbol{C}_0} = \boldsymbol{C}_0(\mathbb{R}^d)$.

Essential part and closure

In the sequel we assume that $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ is a Banach algebra with bounded approximate units, such as $\left(\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_1\right)$ (with convolution), or $\left(\boldsymbol{C}_0(\mathbb{R}^d), \|\cdot\|_\infty\right)$ or $\left(\mathcal{F}\boldsymbol{L}^1(\mathbb{R}^d), \|\cdot\|_{\mathcal{F}\boldsymbol{L}^1}\right)$ with pointwise multiplication.

Theorem

Let $\boldsymbol{A}$ be a Banach algebra with bounded approximate units, and $\boldsymbol{B}$ a Banach module over $\boldsymbol{A}$. Then we have the following general identifications:

$$(\boldsymbol{B}_{\boldsymbol{A}})_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad (\boldsymbol{B}^{\boldsymbol{A}})_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad (\boldsymbol{B}_{\boldsymbol{A}})^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}, \quad (\boldsymbol{B}^{\boldsymbol{A}})^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}. \tag{16}$$

or in a slightly more compact form:

$$\boldsymbol{B}_{\boldsymbol{A}\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad {\boldsymbol{B}^{\boldsymbol{A}}}_{\boldsymbol{A}} = \boldsymbol{B}_{\boldsymbol{A}}, \quad {\boldsymbol{B}_{\boldsymbol{A}}}^{\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}, \quad \boldsymbol{B}^{\boldsymbol{A}\boldsymbol{A}} = \boldsymbol{B}^{\boldsymbol{A}}. \tag{17}$$

Essential Banach modules and BAIs

The usual way to define the *essential part* $\boldsymbol{B_A}$ resp. $\boldsymbol{B_e}$ of a Banach module $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ with respect to some Banach algebra action $(\mathbf{a}, \mathbf{b}) \mapsto \mathbf{a} \bullet \mathbf{b}$ is defined as the closed linear span of $\boldsymbol{A} \bullet \boldsymbol{B}$ within $\left(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}}\right)$. This subspace has other nice characterizations using BAIs (bounded approximate units (BAI) in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$):

Lemma

For any BAI $(\mathbf{e}_\alpha)_{\alpha \in I}$ in $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ one has:

$$\boldsymbol{B_A} = \{\mathbf{b} \in \boldsymbol{B} \mid \lim_\alpha \mathbf{e}_\alpha \bullet \mathbf{b} = \mathbf{b}\} \tag{18}$$

The Cohen-Hewitt Factorization Theorem

In particular one has: Let $(\mathbf{e}_\alpha)_{\alpha \in I}$ and $(\mathbf{u}_b)_{\beta \in J}$ be two bounded approximate units (i.e. bounded nets within $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ acting in the limit like an identity in the Banach algebra $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$. Then

$$\lim_\alpha \mathbf{e}_\alpha \bullet \mathbf{b} = \mathbf{b} \Leftrightarrow \lim_\beta \mathbf{u}_\beta \bullet \mathbf{b} = \mathbf{b}. \tag{19}$$

Theorem

(The Cohen-Hewitt factorization theorem, without proof, see [6]) Let $(\boldsymbol{A}, \|\cdot\|_{\boldsymbol{A}})$ be a Banach algebra with some BAI of size $C > 0$, then the algebra factorizes, which means that for every $a \in \boldsymbol{A}$ there exists a pair $a', h' \in \boldsymbol{A}$ such that $a = h' \cdot a'$, in short: $\boldsymbol{A} = \boldsymbol{A} \cdot \boldsymbol{A}$. In fact, one can even choose $\|a - a'\| \leq \varepsilon$ and $\|h'\| \leq C$.

Essential part and closure II

Having now Banach spaces of distributions which have two module structures, we have to use corresponding symbols. FROM NOW ON we will use the letter $\boldsymbol{A}$ mostly for pointwise Banach algebras and thus for the $\mathcal{F}\boldsymbol{L}^1$-action on $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, and we will use the symbol G (because convolution is coming from the integrated group action!) for the $\boldsymbol{L}^1$ convolution structure. We thus have

$$\boldsymbol{B}_{\boldsymbol{G}\boldsymbol{G}} = \boldsymbol{B}_{\boldsymbol{G}}, \quad {\boldsymbol{B}^{\boldsymbol{G}}}_{\boldsymbol{G}} = \boldsymbol{B}_{\boldsymbol{G}}, \quad \boldsymbol{B}_{\boldsymbol{G}}^{\boldsymbol{G}} = \boldsymbol{B}^{\boldsymbol{G}}, \quad \boldsymbol{B}^{\boldsymbol{G}\boldsymbol{G}} = \boldsymbol{B}^{\boldsymbol{G}}. \tag{20}$$

In this way we can combine the two operators (in view of the above formulas we can call them interior and closure operation) with respect to the two module actions and form spaces such as

$${\boldsymbol{B}^{\boldsymbol{G}}}_{\boldsymbol{A}}, \quad {{\boldsymbol{B}_{\boldsymbol{A}}}^{\boldsymbol{G}}}_{\boldsymbol{A}}, \quad {{{\boldsymbol{B}^{\boldsymbol{G}}}_{\boldsymbol{A}}}^{\boldsymbol{G}}}_{\boldsymbol{A}} \dots$$

or changes of arbitrary length, as long as the symbols $\boldsymbol{A}$ and $\boldsymbol{G}$ appear in alternating form (at any position, upper or lower).

Combining the two module structures

Fortunately one can verify (paper with W.Braun from 1983, J.Funct.Anal.) that any "long" chain can be reduced to a chain of at most two symbols, the *last occurence of each of the two symbols being the relevant one!* So in fact all the three symbols in the above chain describe the same space of distributions.
But still we are left with the follwoing collection of altogether eight two-letter symbols:

$$\boldsymbol{B}_{\boldsymbol{GA}}, \boldsymbol{B}_{\boldsymbol{AG}}, \boldsymbol{B}_{\boldsymbol{A}}{}^{\boldsymbol{G}}, \boldsymbol{B}^{\boldsymbol{G}}{}_{\boldsymbol{A}}, \boldsymbol{B}_{\boldsymbol{G}}{}^{\boldsymbol{A}}, \boldsymbol{B}^{\boldsymbol{A}}{}_{\boldsymbol{G}}, \boldsymbol{B}^{\boldsymbol{AG}}, \boldsymbol{B}^{\boldsymbol{GA}} \tag{21}$$

and of course the four one-symbol objects

$$\boldsymbol{B}_{\boldsymbol{A}}, \boldsymbol{B}_{\boldsymbol{G}}, \boldsymbol{B}^{\boldsymbol{A}}, \boldsymbol{B}^{\boldsymbol{G}} \tag{22}$$

Some structures, simple facts

There are other, quite simple and useful facts, such as

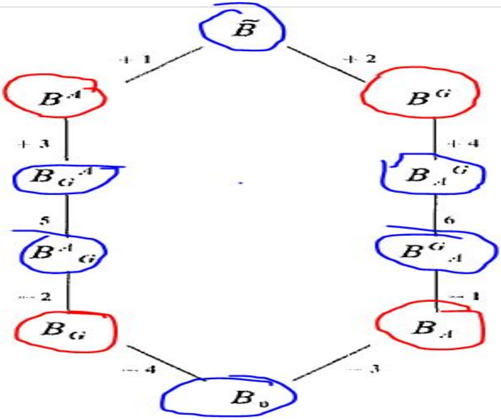
$$\mathcal{H}_{\boldsymbol{A}}({\boldsymbol{B}^1}_{\boldsymbol{A}}, \boldsymbol{B}^2) = \mathcal{H}_{\boldsymbol{A}}({\boldsymbol{B}^1}_{\boldsymbol{A}}, {\boldsymbol{B}^2}_{\boldsymbol{A}}) \tag{23}$$

which can easily be verified if ${\boldsymbol{B}^1}_{\boldsymbol{A}} = \boldsymbol{A} \bullet \boldsymbol{B}^1$, since then $T \in \mathcal{H}_{\boldsymbol{A}}({\boldsymbol{B}^1}_{\boldsymbol{A}}, \boldsymbol{B}^2)$ applied to $\mathbf{b}^1 = \mathbf{a} \bullet \mathbf{b}^{1'}$ gives

$$T(\mathbf{b}^1) = T(\mathbf{a} \bullet \mathbf{b}^{1'}) = \mathbf{a} \bullet T(\mathbf{b}^{1'}) \in {\boldsymbol{B}^2}_{\boldsymbol{A}}.$$

The Main Diagram

This diagram is taken from [1].



Minimal and Maximal Spaces

More or less the diagram shows that for every FouSS norm there exists a whole family of function spaces *with the SAME norm*, i.e. closed within each other.

A FouSS space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called **maximal** if $\boldsymbol{B} = \widetilde{\boldsymbol{B}}$, or equivalently the following is true:
Assume that σ_α is a bounded net of elements in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$, with $\sigma_0 = w^* - lim_\alpha \sigma_a$ (in $\boldsymbol{S}_0'(\mathbb{R}^d)$) implies that $\sigma_0 \in \boldsymbol{B}$.

A FouSS space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ is called **minimal** if $\boldsymbol{B} = \boldsymbol{B}_{AG}$, resp. if $\boldsymbol{S}_0(\mathbb{R}^d)$ is dense in $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$.

Characterization

Theorem

A Banach space $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *is maximal if and only if it is the dual space of some other minimal Fourier standard space. The predual can be determined as*

$$\boldsymbol{B} = (\boldsymbol{B}_o)'_o.$$

Example: $\boldsymbol{B} = L^\infty(\mathbb{R}^d)$ is the dual space. $\boldsymbol{B}_o = \boldsymbol{C}_0(\mathbb{R}^d)$, hence $\boldsymbol{B}'_o = \boldsymbol{M}_b(\mathbb{R}^d)$, and consequently $(\boldsymbol{B}'_o)_o = \boldsymbol{L}^1(\mathbb{R}^d)$ shows up as the pre-dual of $\boldsymbol{L}^\infty(\mathbb{R}^d)$.

Theorem

A FouSS $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ *is reflexive if and only if* $\boldsymbol{B}$ *and* $\boldsymbol{B}'$ *are minimal and maximal.*

Outlook

I hope that I have shown that this view on (classical and modern) function spaces provide still a large number of *interesting and open questions*. So Fourier Analysis is not at all an outdated subject area within mathematical analysis.

We have not even discussed *Time-Frequency Analysis*, Gabor Analysis, wavelet theory, shearlet theory, or the theory of pseudo-differential operators.

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ZUGABE!

`MR2427981 (2009j:46071) Mockenhaupt, G. ; Ricker, W. J. Optimal extension of the Hausdorff-Young inequality. J. Reine Angew. Math. 620 (2008), 195211.`

Main result: For the typical Hausdorff-Young setting, i.e. for $p \in [1,2)$ there is a solid (meaning Banach lattice) FouSS $(\boldsymbol{B}, \|\cdot\|_{\boldsymbol{B}})$ strictly larger than $\boldsymbol{L}^p$ such that still $\mathcal{F}(\boldsymbol{B}) \subset \boldsymbol{L}^q$. This space can be described as the pointwise multipliers from $\boldsymbol{L}^\infty$ to $\mathcal{F}^{-1}(\boldsymbol{L}^p)$, so it is obviously solid (because pointwise multiplication with $\boldsymbol{L}^\infty$ is a bounded operation) and has the right Fourier property. The strict inclusion requires analysis!

Constructions within the FouSS Family IV

There is a small body of literature (mostly papers by Kelly McKennon, a former PhD student of Edwin Hewitt) concerning spaces of "tempered elements". He has done the case starting $\boldsymbol{B} = \boldsymbol{L}^p(G)$, over general LC groups, but the construction makes sense if (and only if) one has a nice invariant space which happens not to be a convolution (or pointwise) algebra.
By *intersecting* the space with its own "multiplier algebra" one obtains an (abstract) Banach algebra, and often the Banach algebra homomorphism of this new algebra "are" just the translation invariant operators on the original spaces.
For the case of $\boldsymbol{B} = \left(\boldsymbol{L}^p(\mathbb{R}^d), \|\cdot\|_p\right)$ one would define

$$\boldsymbol{L}_p^t := \boldsymbol{L}^p \cap \mathcal{H}_G(\boldsymbol{L}^p, \boldsymbol{L}^p).$$

understood as the intersection of two FouSSs, with the natural norm, which is the sum of the $\boldsymbol{L}^p$-norm of f plus the operator norm of the convolution operator.
For $p > 2$ one has to be careful and has to define that operator norm only by looking at the action of $k \to k * f$ on $\boldsymbol{C}_c(\mathbb{R}^d)$! (convolution in the pointwise sense might fail to exist, on more than just a null-set!).
However it is not a problem to approximate every element (in norm or even just in the w^*-sense by test-functions in $\boldsymbol{S}_0(\mathbb{R}^d)$ and then take the limit of the convolution products of the regularized expressions.

A current open question (first disclosure)

It is clear that $\mathcal{S}(\mathbb{R})$ is a (dense) subspace of $\left(\mathbf{S}_0(\mathbb{R}), \|\cdot\|_{\mathbf{S}_0}\right)$. Consequently each of the *Hermite functions* $(h_n)_{n\geq 0}$ belongs to $\mathbf{S}_0(\mathbb{R})$, and one may ask the following questions

1. Is the sequence $(h_n)_{n\geq 0}$ *bounded* in $\left(\mathbf{S}_0(\mathbb{R}), \|\cdot\|_{\mathbf{S}_0}\right)$?
2. Are the projection operators on the Hermite functions

$$P_n : f \to \langle f, h_n \rangle h_n$$

uniformly bounded on $\left(\mathbf{S}_0(\mathbb{R}), \|\cdot\|_{\mathbf{S}_0}\right)$?

3. Are the Hermite partial sums

$$Q_N : f \to \sum_{n=0}^{N} \langle f, h_n \rangle h_n$$

uniformly bounded on $\mathbf{S}_0(\mathbb{R})$??

Partial answers to these questions

In fact: the sequence of Hermite functions is *unbounded* in $\boldsymbol{S}_0(\mathbb{R})$, in fact their norms grow like $n^{1/4}$ (cf. work by A.J.E.M. Janssen).

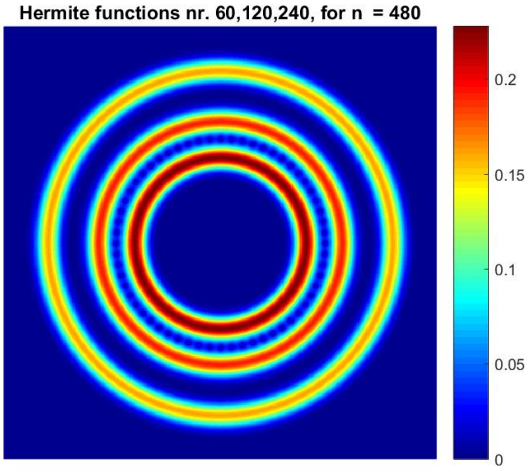
However, their norm in $(\boldsymbol{S}_0'(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0'})$ is decaying at the same rate, i.e. we have

$$\|h_n\|_{\boldsymbol{S}_0'} \approx n^{-1/4}, \quad n \to \infty.$$

Consequently the projection operators P_n are uniformly bounded on $\left(\boldsymbol{S}_0(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0}\right)$ (and being symmetric also on $(\boldsymbol{S}_0'(\mathbb{R}), \|\cdot\|_{\boldsymbol{S}_0'})$).

It is thus also clear that absolute convergence of the Hermite expansions does not define a FouSS, although it is obviously a nice, Fourier invariant function space. Corresponding weighted ℓ^2-conditions give rise to the so-called *Shubin classes* $\boldsymbol{Q}_s(\mathbb{R}^d)$.

The TF-content of three Hermite functions



THANKS for your ATTENTION!

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