Nikol'skii inequality between the uniform norm and integral q-norm with the Bessel weight on the semi-axis for entire functions of exponential type

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Statement of the problem

Let $L^q_\alpha = L^q\left([0,\infty), x^{2\alpha+1}\right)$ with $1 \le q < \infty$ and $\alpha > -1$ be the set of complex-valued Lebesgue measurable functions f on the semi-axis $\mathbb{R}_+ = [0,\infty)$ such that the product $|f(x)|^q\, x^{2\alpha+1}$ is integrable over $\mathbb{R}_+$. The space L^q_α is endowed with the norm

$$\|f\|_{q,\alpha} = \|f\|_{L^q_\alpha} = \left(\int_0^\infty |f(x)|^q\, x^{2\alpha+1} dx\right)^{1/q}, \quad f \in L^q_\alpha.$$

In the case $q = \infty$ $(\alpha > -1)$, we assume that L^∞_α is the space $L^\infty = L^\infty(0,\infty)$ of functions f measurable and essentially bounded on $\mathbb{R}_+$. This space is endowed with the norm

$$\|f\|_\infty = \operatorname{ess\,sup}\{|f(x)| \colon x \in (0,\infty)\}, \quad f \in L^\infty.$$

Along with L^∞, consider the space $C = C[0,\infty)$ of functions continuous and bounded on $\mathbb{R}_+$ with the uniform norm

$$\|f\|_{C[0,\infty)} = \max\{|f(x)| \colon x \in [0,\infty)\}.$$

Statement of the problem

Denote by $\mathfrak{W}(\sigma, q, \alpha)$ the set of even entire functions of exponential type (at most) $\sigma > 0$, the restriction of which to the semi-axis $[0, \infty)$ belongs to the space L^q_α.
S.S.Platonov [Platonov-2007] performed an in-depth study of approximative and extremal properties of the class $\mathfrak{W}(\sigma, q, \alpha)$ in the space L^q_α.
In particular, he proved that, for $1 \le q < p \le \infty$ and $\alpha > -1/2$, the following Nikol'skii type inequality holds for functions of the class $\mathfrak{W}(\sigma, q, \alpha)$:

$$\|f\|_{p,\alpha} \le C\,\sigma^{(2\alpha+2)(1/q-1/p)}\,\|f\|_{q,\alpha}, \quad f \in \mathfrak{W}(\sigma, q, \alpha), \tag{1}$$

with some constant $C = C(q, p, \alpha)$ (see [Platonov-2007, Theorem 3.5], this result was announced earlier in [Platonov-2004]).

Statement of the problem

We will discuss inequality (1) for $p = \infty$, i.e., the inequality

$$\|f\|_{C[0,\infty)} \leq M\,\|f\|_{q,\alpha}, \quad f \in \mathfrak{W}(\sigma, q, \alpha), \tag{2}$$

with the best (least possible) constant $M = M(\sigma, \alpha, q)$. The dependence of $M = M(\sigma, \alpha, q)$ on the parameter σ is known; more precisely,

$$M = M(\sigma, \alpha, q) = M_0(\alpha, q)\,\sigma^{(2\alpha+2)(1/q-1/p)}.$$

Our aim is to study extremal functions of inequality (2), i.e., functions $\rho_\sigma \in \mathfrak{W}(\sigma, q, \alpha)$, $\rho_\sigma \not\equiv 0$, at which the inequality becomes an equality. In particular, we study the property of uniqueness of extremal functions. It is clear that, if a function ρ_σ is extremal, then the function $c\rho_\sigma$ for any constant $c \neq 0$ is also extremal. If ρ_σ is an extremal function in inequality (2) and every extremal function has the form $c\,\rho_\sigma,\ c \in \mathbb{C}$, then ρ_σ is said to be the *unique* extremal function of inequality (2).

Statement of the problem

Extremal (and especially approximative) properties of entire functions of exponential type of one and many variables is a large part of function theory. Such problems were studied by S.N.Bernstein, B.M.Levitan, B.Ya.Levin, N.I.Akhiezer, S.M.Nikol'skii, S.S.Platonov, Q.I.Rahman, G.Schmeisser, D.V.Gorbachev, O.L.Vinogradov, A.V.Gladkaya, S.Yu.Tikhonov, and others.

Even more extensive is the related topic of extremal properties of algebraic polynomials on an interval, domains of the complex plane, the Euclidean sphere, and other manifolds and trigonometric polynomials in one and several variables.

In what follows, we only refer to the results closely related to the subject of our study.

Statement of the problem

If $\alpha = \frac{n}{2} - 1$, where n is a nonnegative integer, then the space L^q_α is isometric to the subspace of spherically symmetrical functions from the space $L^q(\mathbb{R}^n)$. Similarly, the space $\mathfrak{W}(\sigma, q, \alpha)$ is related to the space of entire functions of n (complex) variables of exponential spherical type σ. Thus, for $\alpha = \frac{n}{2} - 1, \; n \in \mathbb{N}$, inequality (1) and, in particular, (2), is contained in Theorem 3.3.5 of Nikol'skii's monograph [Nik-1977].

The inequality

$$|f(0)| \le D\,\|f\|_{q,\alpha}, \qquad f \in \mathfrak{W}(\sigma, q, \alpha) \tag{3}$$

with the best constant $D = D(\sigma, q, \alpha)$, which is related to (2), plays an important role in what follows.

It is clear that $D \le M$. Actually, at least for $\alpha > -1/2$, we have the equality $D = M$.

Entire functions that deviate least from zero

Consider the set

$$\mathfrak{W}[1](\sigma, q, \alpha) = \{f \in \mathfrak{W}(\sigma, q, \alpha) \colon f(0) = 1\} \tag{4}$$

of entire functions from $\mathfrak{W}(\sigma, q, \alpha)$ equal to 1 at the point 0. Let

$$\Delta = \inf\{\|f\|_{q,\alpha} \colon f \in \mathfrak{W}[1](\sigma, q, \alpha)\}. \tag{5}$$

It is clear that $D = 1/\Delta$.
Thus, the problem on sharp inequality (3) coincides with problem (5) on the least deviation form zero of the class (4) of entire functions.
Problems on entire function that deviate least from zero were studied by S.N.Bernstein, N.I.Akhiezer,O.L.Vinogradov, A.V.Gladkaya, and others.
However, these problems are much less studied as compared with similar problems for algebraic and trigonometric polynomials.

Value (5) can be interpreted as the best approximation in the space L^q_α of an arbitrary function from set (4) by the subset

$$\mathfrak{W}[0](\sigma, q, \alpha) = \{f \in \mathfrak{W}(\sigma, q, \alpha) \colon f(0) = 0\} \tag{6}$$

of functions from $\mathfrak{W}(\sigma, q, \alpha)$ vanishing at the point 0.
Therefore, it is reasonable to expect that the following statement is valid.

Theorem 1

For $1 \le q < \infty,\ \alpha > -1$, *and* $\sigma > 0$, *an extremal function* $\varrho_\sigma = \varrho_{\sigma,q,\alpha} \in \mathfrak{W}(\sigma, q, \alpha),\ \ \varrho_\sigma \not\equiv 0$ *in inequality* (3) *exists and its characteristic property is the "orthogonality" to set* (6)*:*

$$\int_0^\infty f(x)\, x^{2\alpha+1} |\varrho_\sigma(x)|^{q-1} \operatorname{sign} \varrho_\sigma(x) dx = 0, \quad f \in \mathfrak{W}[0](\sigma, q, \alpha). \qquad (7)$$

For $1 < q < \infty$, *an extremal function in inequality* (3) *is unique.*

By now, our attempts to prove the uniqueness of an extremal function in inequality (3) for $q = 1$ were not successful.

Main result

Theorem 2

For $\alpha > -1/2,\ 1 \leq q < \infty$, and $\sigma > 0$, the following statements are valid.
(1) *The best constants in inequalities (2) and (3) coinside:*

$$M(\sigma, q, \alpha) = D(\sigma, q, \alpha).$$

(2) *Inequalities (2) and (3) have the same set of extremal functions. A characteristic property of an extremal function $\varrho_{\sigma,q,\alpha}$ in inequalities (2) and (3) is (7). For $1 < q < \infty$, this function is unique.*
(3) *Any extremal function in inequality (2) attains its uniform norm on the semi-axis $[0, \infty)$ only at the point $x = 0$.*

Main result

An essential step in the proof of Theorem 2 is to prove that an extremal function in inequality (2) attains its uniform norm only at the end-point 0 of the semi-axis $[0, \infty)$. To prove this fact, we apply the Bessel generalized translation operator.

We need to know the norm of the generalized translation in the space L_α^q and to study the attainability of the norm.

Bessel functions

The Bessel functions J_α (of the first kind) of order α are important in mathematics and it applications. The literature on their properties is quite extensive (see monographs [Watson-1945] and [Bateman-Erdeji-1953] and the textbook [Gray-Mathews]). The Bessel function is defined by the formula

$$J_\alpha(z) = \left(\frac{z}{2}\right)^\alpha \sum_{k=0}^{\infty} \frac{(-1)^k}{\Gamma(k+1)\Gamma(k+\alpha+1)} \left(\frac{z}{2}\right)^{2k}. \tag{8}$$

By the D'Alembert test, the series on the right-hand side of (8) converges (absolutely) everywhere in the complex plane $\mathbb{C}$; hence, its sum is an entire function with nonzero value at the point 0.

Consequently, if α is a (nonnegative) integer, then J_α is a single-valued analytic function. For noninteger values of α, the function J_α is multivalued; this function is defined everywhere in the complex plane in the case $\alpha \geq 0$, and everywhere except the point 0 in the case $\alpha < 0$.

The normalized Bessel function

$$j_\alpha(z) = \Gamma(\alpha+1)\left(\frac{2}{z}\right)^\alpha J_\alpha(z)$$

plays an important role. According to (8), the function j_α is the sum of the series

$$j_\alpha(z) = \sum_{k=0}^{\infty} \frac{(-1)^k \Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(k+\alpha+1)} \left(\frac{z}{2}\right)^{2k}. \tag{9}$$

Series (9) converges in the entire complex plane $\mathbb{C}$; therefore, function (9) is entire. In particular, we have

$$j_{-\frac{1}{2}}(z) = \cos z, \quad j_{\frac{1}{2}}(z) = \frac{\sin z}{z}, \quad j_{\frac{3}{2}}(z) = \frac{3}{z^2}\left(\frac{\sin z}{z} - \cos z\right). \tag{10}$$

For $\alpha > -1/2$, the function j_α can be represented in the form of Poisson integral (see, for example, [Watson-1945, Ch. III, Sect. 3.3, (1)])

$$j_\alpha(z) = \frac{\Gamma(\alpha+1)}{h_\alpha} \int_0^1 (1-t^2)^{\alpha-\frac{1}{2}} \cos(zt)dt. \tag{11}$$

For $\alpha \geq -\frac{1}{2}$, the following inequality holds [Watson-1945, Ch. III, Sect. 3.31, (1)]:

$$|j_\alpha(z)| \leq e^{|\mathrm{Im} z|}, \qquad z \in \mathbb{C}. \tag{12}$$

For $\alpha > -1/2$, inequality (12) follows from (11); for $\alpha = -1/2$ it follows from the explicit form of the function $j_{-1/2}$ and (10).

Estimate (12) implies that the (entire) function j_α has exponential type 1.

The functions j_α have the following properties:

$$|j_\alpha(t)| \le j_\alpha(0) = 1, \quad \alpha \ge -\frac{1}{2}, \quad t \in \mathbb{R}; \tag{13}$$

$$\lim_{u\to\infty} j_\alpha(u) = 0, \quad \alpha > -\frac{1}{2}. \tag{14}$$

Property (13) can be found in [Watson-1945, Ch. III, Sects. 3.3, 3.31], [Bateman-Erdeji-1953, Ch. 7, Sect. 7.3, (4)]. Property (14) follows from known asymptotic expansions of $J_\alpha(z)$ as $z \to \infty$ [Watson-1945, Ch. VIII, Sect. 7.21], [Bateman-Erdeji-1953, Ch. 7, Sect. 7.13]. It is also not hard to prove the latter property using representation (11).

Generalized Bessel translation

The Bessel generalized translation operator with step $t \in [0, \infty)$ for $\alpha > -1/2$ is said to be the operator

$$T_t f(x) = T_t^\alpha f(x) = \gamma(\alpha) \int_0^\pi f\left(\sqrt{t^2 + x^2 - 2xt\cos\varphi}\right) \sin^{2\alpha}\varphi d\varphi; \quad (15)$$

where,

$$\gamma(\alpha) = \frac{\Gamma(\alpha+1)}{\Gamma\left(\frac{1}{2}\right)\Gamma\left(\alpha+\frac{1}{2}\right)} = \frac{1}{\displaystyle\int_0^\pi \sin^{2\alpha}\varphi d\varphi}.$$

The translation operator (15) is generated by the identity

$$T_t \eta_y(x) = \eta_y(t)\eta_y(x), \quad t, x \geq 0, \quad (16)$$

for functions $\eta_y(x) = j_\alpha(yx)$ depending on the parameter $y \geq 0$; identity (16) is called a product formula (for Bessel functions (8)). The product formula (16) was first obtained by L.Gegenbauer in 1875 [Watson-1945, Sect. 11.41, (16)].

Bessel translation

Properties of the generalized translation operator were studied in-depth by B.M.Levitan [Levitan-1951]. In particular, he proved [Levitan-1951, p. 125] that, for all $\alpha \geq -1/2$, the operator T_t is self-adjoint. More precisely, if a function $f \in L^1_\alpha$ is continuous and $g \in C[0,\infty)$, then we have

$$\int_0^\infty (T_t f)(x) g(x) x^{2\nu+1} dx = \int_0^\infty f(x)(T_t g)(x) x^{2\nu+1} dx.$$

The generalized translation operator finds important applications in mathematics, in particular, in approximation theory, where by means of the operator T_t the smoothness of functions is given; see, in particular, [Babenko-1998], [Platonov-2007], and the references therein.

Bessel translation

There are several approaches (with equivalent results) of defining and studying the generalized translation operator T_t in the spaces L^q_α. One of these approaches is to define the operator T_t and obtain its desired properties on a class of smooth function dense in the space L^q_α and extend the operator by continuity to the entire L^q_α. For example, in [Platonov-2007], this was performed by means of the space $\mathcal{S}_+$ of even infinitely differentiable functions on the axis vanishing at infinity together with their derivatives of any order faster than the absolute value of their argument to any power. In [Platonov-2007], it is proved that, for all $\alpha > -1/2$, $1 \leq q \leq \infty$, and $t \geq 0$, the operator T_t is a bounded linear operator in L^q_α; moreover,

$$\|T_t\|_{q,\alpha} = \|T_t\|_{L^q_\alpha \to L^q_\alpha} \leq 1.$$

It follows from (16) and (13) that the equality holds:

$$\|T_t\|_{q,\alpha} = 1. \qquad (17)$$

In addition to statement (17), we need to know, whether the norm of the operator T_t for $t > 0$ in L^q_α is attained. For the beginning, let us transform relation (15) for the operator T_t in the space $C[0,\infty)$. For $f \in C[0,\infty)$ and $xt > 0$, we have

$$T_t f(x) = \int_{|x-t|}^{x+t} f(u)\, F(t,x,u)\, du,$$

where

$$F(t,x,u) =$$

$$= \gamma(\alpha) \left(\sqrt{(u^2 - (x-t)^2)\,((x+t)^2 - u^2)} \right)^{2\alpha-1} \frac{2u}{(2xt)^{2\alpha}}.$$

The function $F(t,x,u)$ is positive with respect to the variable $u \in (|x-t|, x+t)$ and

$$\int_{|x-t|}^{x+t} F(t,x,u)\, du = 1.$$

Generalized translation operator in $L^1\left([0,\infty), t^{2\alpha+1}\right)$

Lemma

For $\alpha > -1/2,\ t > 0$, and $q = 1$, the norm of the operator T_t in the space L^1_α is attained at a function $f \in L^1\left([0,\infty), t^{2\alpha+1}\right)$ nonzero almost everywhere on $(0,\infty)$ if and only if the function f maintains sign (almost everywhere) on $(0,\infty)$.

Generalized translation operator in $L^q\left([0,\infty), t^{2\alpha+1}\right),\ 1 < q < \infty$

Lemma

For $\alpha > -1/2,\ t > 0$, and $1 < q < \infty$, the norm of the operator T_t in the space L^q_α is not attained.

The first step of the proof of the main theorem

For the best constants in inequalities (2) and (3) the inequality $D \le M$ is valid. Let us show that in fact they coincide:

$$D = M. \tag{18}$$

We will use the generalized translation operator (15). Let $f \in \mathfrak{W}(\sigma, q, \alpha)$. It is known that $f(t) \to 0$ as $t \to \infty$. Therefore, there exists a point $t = t(f) \in [0, \infty)$, at which the uniform norm of the function f on the semi-axis $[0, \infty)$ is attained. The function

$$g(x) = (T_t f)(x), \quad x \in [0, \infty),$$

also belongs to the class $\mathfrak{W}(\sigma, q, \alpha)$ and has the property $g(0) = f(t)$.

Applying inequality (3) and the property

$$\|T_t\|_{q,\alpha} = 1,$$

we obtain

$$\|f\|_{C[0,\infty)} = |f(t)| = |g(0)| \leq D\|g\|_{L^q_\alpha} \leq D\|f\|_{L^q_\alpha}.$$

Thus, $\|f\|_C \leq D\|f\|_{L^q_\alpha}$. Since $f \in \mathfrak{W}(\sigma, q, \alpha)$ is arbitrary, this implies the inequality $M \leq D$. Relation (18) is verified.

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Thank you for your attention!