Weighted norm inequalities for integral transforms with kernels bounded by power functions

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Brief history: weighted norm inequalities for the Fourier transform

For the Fourier transform

$$\widehat{f}(y) = \int_{\mathbb{R}} f(x)e^{-ixy}\,dx,$$

it was proved in the 1980s (Muckenhoupt, Jurkat-Sampson) that the weighted norm inequality

$$\|\widehat{f}\|_{q,u} := \left(\int_{\mathbb{R}} u(y)|\widehat{f}(y)|^q\,dy\right)^{1/q} \leq C\left(\int_{\mathbb{R}} v(x)|f(x)|^p\,dx\right)^{1/p} =: \|f\|_{p,v}$$

holds for every f with $1 < p \leq q < \infty$ and $C > 0$ independent of f provided that there exists $D > 0$ such that for every $r > 0$

$$\left(\int_0^{1/r} u^*(y)\,dy\right)^{1/q}\left(\int_0^r (1/v)^*(x)^{1-p'}\,dx\right)^{1/p'} \leq D.$$

Problem

Given an integral transform

$$Ff(y) = y^{c_0} \int_0^\infty x^{b_0} f(x) K(x,y)\, dx, \qquad y > 0, \quad b_0, c_0 \in \mathbb{R},$$

where

$$|K(x,y)| \lesssim \min\left\{(xy)^{b_1}, (xy)^{b_2}\right\}, \qquad b_1 > b_2.$$

We also assume $\int_0^1 x^{b_0+b_1}|f(x)|\, dx + \int_1^\infty x^{b_0+b_2}|f(x)|\, dx < \infty$.
We want to give sufficient (and necessary, when possible) conditions for the weighted norm inequality

$$\|y^{-\beta} Ff\|_q \le C\|x^\gamma f\|_p, \quad 1 < p \le q < \infty, \tag{1}$$

to hold, using an approach that does not involve decreasing rearrangements.

Examples

1. The Fourier transform is **not** a good example, since $|K(x,y)| = |e^{2\pi i xy}| = 1$ does not satisfy

$$|K(x,y)| \lesssim \min\left\{(xy)^{b_1}, (xy)^{b_2}\right\}, \quad b_1 > b_2.$$

   The cosine transform is also a bad example.

2. The sine transform satisfies

$$|K(x,y)| = |\sin xy| \lesssim \min\left\{xy, 1\right\},$$

   i.e., $b_1 = 1 > 0 = b_2$.

Examples: The Hankel transform

3. The classical Hankel transform of order $\alpha \geq -1/2$ is defined as

$$H_\alpha f(y) = \int_0^\infty x^{2\alpha+1} f(x) j_\alpha(xy)\, dx,$$

where j_α is the *normalized Bessel function of order* α, represented through the series

$$j_\alpha(z) = \Gamma(\alpha+1) \sum_{n=0}^\infty \frac{(-1)^n (z/2)^{2n}}{n!\Gamma(n+\alpha+1)}.$$

The function j_α satisfies the estimate

$$|j_\alpha(xy)| \lesssim \min\left\{1, (xy)^{-\alpha-1/2}\right\}.$$

4. The so-called $\mathscr{H}_\alpha$ transform is defined as

$$\mathscr{H}_\alpha f(y) = \int_0^\infty (xy)^{1/2} f(x) \mathbf{H}_\alpha(xy)\, dx, \quad \alpha > -1/2,$$

where $\mathbf{H}_\alpha$ is the *Struve function*, defined as

$$\mathbf{H}_\alpha(z) = \left(\frac{z}{2}\right)^{\alpha+1} \sum_{n=0}^{\infty} \frac{(-1)^n (z/2)^{2n}}{\Gamma(n+3/2)\Gamma(n+\alpha+3/2)}.$$

Examples: The $\mathscr{H}_\alpha$ transform

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The Struve function satisfies the estimate

$$|\mathbf{H}_\alpha(x)| \lesssim \begin{cases} \min\{x^{\alpha+1}, x^{-1/2}\}, & \alpha < 1/2, \\ \min\{x^{\alpha+1}, x^{\alpha-1}\}, & \alpha \geq 1/2, \end{cases}$$

- P. G. Rooney, Canad. J. Math (1980).

Known results

- Cosine transform (and Fourier transform): if $Ff = \widehat{f}$ or $Ff = \widehat{f}_{\cos}$, then (1) holds **if and only if** $\beta = \gamma + 1/q - 1/p'$ and

$$\max\{1/q - 1/p', 0\} \leq \beta < 1/q.$$

- W. B. Jurkat–G. Sampson, Indiana Univ. Math. J. (1984); B. Muckenhoupt, Proc. Amer. Math. Soc. (1983).
- H. P. Heinig, Indiana Univ. Math. J. (1984).

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- Sine transform: if $Ff = \widehat{f}_{\sin}$, (1) holds **if and only if** $\beta = \gamma + 1/q - 1/p'$ and

$$\max\{1/q - 1/p', 0\} \leq \beta < 1 + 1/q.$$

  - D. Gorbachev, E. Liflyand, S. Tikhonov, Indiana Univ. Math. J. (to appear).

Known results

- Hankel transform: if $Ff = H_\alpha f$ $(\alpha \geq -1/2)$, then (1) holds **if and only if** $\beta = \gamma - 2\alpha - 1 + 1/q - 1/p'$ and

$$\max\{1/q - 1/p', 0\} - \alpha - 1/2 \leq \beta < 1/q.$$

- P. Heywood, P. G. Rooney, Proc. Roy. Soc. Edinburgh (1984).
- L. de Carli, J. Math. Anal. Appl. (2008).

Known results

- $\mathscr{H}_\alpha$ transform: if $Ff = \mathscr{H}_\alpha f$ $(\alpha > -1/2)$, (1) holds **provided that** $\beta = \gamma + 1/q - 1/p'$ and
  - for $-1/2 < \alpha < 1/2$, $\beta \geq \max\{1/q - 1/p', 0\}$ and

$$1/q + \alpha - 1/2 < \beta < 1/q + \alpha + 3/2;$$

  - for $\alpha \geq 1/2$,

$$1/q + \alpha - 1/2 < \beta < 1/q + \alpha + 3/2.$$

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Main result (sufficient conditions)

The following states sufficient conditions for inequality (1) to hold.

Theorem

Let $1 < p \le q < \infty$. *If the integral transform*

$$Ff(y) = y^{c_0} \int_0^\infty x^{b_0} f(x) K(x,y)\, dx, \qquad y > 0, \quad b_0, c_0 \in \mathbb{R},$$

satisfies $|K(x,y)| \lesssim \min\left\{(xy)^{b_1}, (xy)^{b_2}\right\}$, *with* $b_1 > b_2$, *then the inequality*

$$\|y^{-\beta} Ff\|_q \le C \|x^\gamma f\|_p, \qquad 1 < p \le q < \infty,$$

holds for every f, *provided that*

$$\beta = \gamma + c_0 - b_0 + \frac{1}{q} - \frac{1}{p'}, \qquad \frac{1}{q} + c_0 + b_2 < \beta < \frac{1}{q} + c_0 + b_1.$$

Sharpness and necessity conditions

If instead of $|K(x,y)| \lesssim \min\left\{(xy)^{b_1},(xy)^{b_2}\right\}$, there holds $K(x,y) \asymp \min\left\{(xy)^{b_1},(xy)^{b_2}\right\}$, the latter theorem can be improved.

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satisfies $|K(x,y)| \lesssim \min\left\{(xy)^{b_1},(xy)^{b_2}\right\}$, with $b_1 > b_2$, then the inequality

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holds for every f with

$$\beta = \gamma + c_0 - b_0 + \frac{1}{q} - \frac{1}{p'}, \qquad \frac{1}{q} + c_0 + b_2 < \beta < \frac{1}{q} + c_0 + b_1.$$

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holds for every f if and only if

$$\beta = \gamma + c_0 - b_0 + \frac{1}{q} - \frac{1}{p'}, \qquad \frac{1}{q} + c_0 + b_2 < \beta < \frac{1}{q} + c_0 + b_1.$$

Examples

We can get the following sufficient conditions for (1):

- Cosine transform: no sufficient conditions! ($b_1 \not> b_2$).

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- Sine transform:

$$1/q < \beta < 1 + 1/q$$
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- Sine transform:

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$$\max\{1/q - 1/p', 0\} \leq \beta < 1 + 1/q.$$

- Hankel transform of order $\alpha > -1/2$:

$$1/q - \alpha - 1/2 < \beta < 1/q$$
$$\max\{1/q - 1/p', 0\} - \alpha - 1/2 \leq \beta < 1/q.$$

Examples

- $\mathscr{H}_\alpha$ transform ($\alpha > -1/2$):

$$\begin{aligned} 1/q < \beta < 1/q + \alpha + 3/2, &\qquad -1/2 < \alpha < 1/2, \\ 1/q + \alpha - 1/2 < \beta < 1/q + \alpha + 3/2, &\qquad \alpha \geq 1/2. \end{aligned}$$

Recall the known sufficient conditions:

– For $-1/2 < \alpha < 1/2$, $\beta \geq \max\{1/q - 1/p', 0\}$ and

$$1/q + \alpha - 1/2 < \beta < 1/q + \alpha + 3/2.$$

– For $\alpha \geq 1/2$,

$$1/q + \alpha - 1/2 < \beta < 1/q + \alpha + 3/2.$$

Transforms with kernel represented by power series

Recall that

$$\|y^{-\beta}\widehat{f}\|_q \leq C\|x^\gamma f\|_p \tag{2}$$

holds for every f if and only if $\beta = \gamma + 1/q - 1/p'$ and

$$\max\{1/q - 1/p', 0\} \leq \beta < 1/q. \tag{3}$$

It was proved by Sadosky and Wheeden that if $\int_{\mathbb{R}} f = 0$, then (2) holds for $1/q < \beta < 1 + 1/q$, additionally to (3).

- C. Sadosky and R. L. Wheeden, Trans. Amer. Mat. Soc. (1987).

Transforms with kernel represented by power series

Assume that the kernel of the transform F is expressed as

$$K(x,y) = (xy)^{b_1} \sum_{m=0}^{\infty} a_m (xy)^{km}, \quad k \in \mathbb{N}, \quad a_m \in \mathbb{C}.$$

If

$$|K(x,y)| \lesssim \min\left\{(xy)^{b_1}, (xy)^{b_2}\right\},$$

with $b_1 \geq b_2$, we write

$$Ff(y) = y^{c_0} \int_0^\infty x^{b_0} (xy)^{b_1} f(x) \big((xy)^{-b_1} K(x,y)\big)\, dx,$$

It is clear that $(xy)^{-b_1}|K(x,y)| \lesssim \min\left\{1, (xy)^{b_2-b_1}\right\}$.

Transforms with kernel represented by power series

If f is a function such that

$$\int_0^\infty x^{b_0+b_1+k\ell} f(x)\,dx = 0, \qquad \ell = 0,\dots,n-1, \quad n \in \mathbb{N}, \tag{4}$$

then we can write

$$Ff(y) = y^{c_0}\int_0^\infty x^{b_0}(xy)^{b_1} f(x)\Bigg((xy)^{-b_1}K(x,y) - \sum_{\ell=0}^{m-1} a_\ell (xy)^{k\ell}\Bigg)\,dx,$$

for any $m = 1,\dots,n$.

Transforms with kernel represented by power series

Since

$$(xy)^{-b_1}|K(x,y)| \lesssim \min\left\{1, (xy)^{b_2-b_1}\right\},$$

with $b_1 \geq b_2$, if we define

$$\mathcal{G}_\ell(x,y) := (xy)^{-b_1}K(x,y) - \sum_{m=0}^{\ell-1} a_m (xy)^{km},$$

the following estimate holds:

$$|\mathcal{G}_\ell(x,y)| \lesssim \min\left\{(xy)^{k\ell}, (xy)^{k(\ell-1)}\right\}.$$

In view of the latter estimate, we can apply our main result.

Transforms with kernel represented by power series

Theorem

Let $1 < p \leq q < \infty$ *and assume the kernel* K *is given by*

$$K(x,y) = (xy)^{b_1} \sum_{m=0}^{\infty} a_m (xy)^{km}, \quad k \in \mathbb{N}, \quad a_m \in \mathbb{C}.$$

Furthermore, suppose that $|K(x,y)| \lesssim \min\left\{(xy)^{b_1}, (xy)^{b_2}\right\}$, *with* $b_1 \geq b_2$. *Then, the weighted norm inequality*

$$\|y^{-\beta} Ff\|_q \lesssim \|x^\gamma f\|_p$$

holds for all f *satisfying* (4) *with* $\beta = \gamma + c_0 - b_0 + 1/q - 1/p'$ *and*

$$1/q + c_0 + b_1 < \beta < 1/q + c_0 + b_1 + n\ell,$$

where $\beta \neq 1/q + c_0 + b_1 + k\ell$ *for* $\ell = 1, \ldots, n-1$.

Application

As a simple example of the latter, consider f with $\int_0^\infty f(x)\,dx = 0$. Since

$$\cos xy = \sum_{n=0}^{\infty} \frac{(-1)^n (xy)^{2n}}{(2n)!},$$

we have that

$$\|y^{-\beta} \widehat{f}_{\cos}\|_q \leq C\|x^\gamma f\|_p, \qquad 1 < p \leq q < \infty,$$

holds with $\beta = \gamma + 1/q - 1/p'$ and

$$\max\{1/q - 1/p', 0\} \leq \beta < 2 + 1/q, \qquad \beta \neq 1/q.$$

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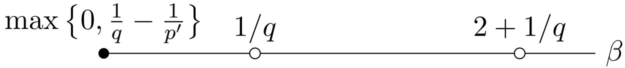

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Thank you!