Spectral decay of finite Fourier transforms and related random matrices.

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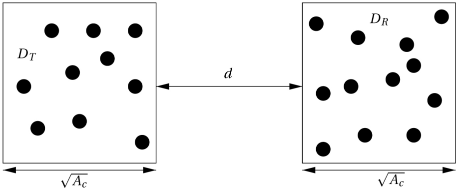
Transmission in a wireless MIMO network.

Considered by Desgroseilliers, Lévêque and Preissmann, EPFL (2013) :



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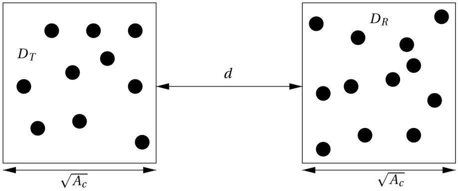

Matrix used to describe the *phase fading* between the emission and the reception :

$$a_{jk} = \frac{e^{2i\pi r_{jk}/\lambda}}{r_{jk}}.$$

Here r_{jk} is the distance between the nodes j and k, which are chosen randomly in two squares and at large distance.

Their questions on the channel fading matrix.

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- Number of degrees of freedom, that is, the number of significative singular values.
- For a value p, which is the total power of the input signal, uniformly distributed among the n nodes, want to have an approximation of the Shannon capacity of the system, given by

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Nodes are chosen randomly. We want to know this with large probability.

The mathematical question.

After approximation, simplification and normalization, the same authors reduce to the study of $n \times n$ matrices

$$A := \frac{\sqrt{m}}{n} \begin{pmatrix} e^{2i\pi m Z_1 Y_1} & \dots & e^{2i\pi m Z_1 Y_n} \\ \vdots & & \vdots \\ e^{2i\pi m Z_n Y_1} & \dots & e^{2i\pi m Z_n Y_n} \end{pmatrix}.$$

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Here Y_j, Z_k independent and uniformly distributed in $I := (-1/2, +1/2)$.
So A is a random matrix, with alea given by the Z_j's and alea given by the Y_k's.

By assumption $m \ll n$ is large.
Describe the singular values of A (i.e. the spectrum of A^*A or AA^*) for m large, m/n small.
Typically, $m \approx n^\delta$, with $1/2 \leq \delta < 1$.

Finite Fourier Transform.

Kind of discretization of the operator on $L^2(I)$ given by

$$\mathcal{F}_m(f)(z) := \sqrt{m} \int_{-1/2}^{+1/2} e^{2i\pi mzy} f(y)dy.$$

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The compact operator $\mathcal{F}_m^*\mathcal{F}_m$ is given by

$$\mathcal{Q}_m(f)(x) = \mathcal{F}_m^*\mathcal{F}_m(f)(x) = \int_{-1/2}^{+1/2} \frac{\sin m\pi(x-y)}{\pi(x-y)} f(y)dy.$$

$\mathcal{Q}_m$ is the *time frequency limiting operator.*
$\frac{\sin \pi x}{\pi x}$ *is known as the sinc kernel.*

Eigenvalues of $\mathcal{Q}_m$ are ordered in a decreasing sequence, starting from 0.

The main theorem for random matrices.

Theorem. The singular values of A are close to the singular values of $\mathcal{F}_m$. More precisely, with probability $1 - 4e^2 e^{-\xi^2/2}$,

$$\left(\sum_{j \geq 0} (\lambda_j(A)^2 - \lambda_j(\mathcal{F}_m)^2)^2 \right)^{1/2} \leq \frac{(2\xi + 1.5)m}{\sqrt{n}}.$$

$\sum_j \lambda_j(A)^2 = \sum_j \lambda_j(\mathcal{F}_m)^2 = m$. So the right hand side is an error if $\frac{m}{n}$ is small.

Notation : $\lambda(M) = \{\lambda_0(M), \cdots \lambda_j(M), \cdots\}$ is the spectrum (or sequence of singular values) of a symmetric positive semi-definite matrix (or symmetric positive semi-definite operator), in decreasing order.

The asymptotic behaviour of $\lambda_j(\mathcal{Q}_m)$.

(Landau and Widom 1980) m tending to ∞ :
$N(\alpha) = \#\{\lambda_j(\mathcal{Q}_m);\ \lambda_j(\mathcal{Q}_m) > \alpha\}$ is such that

$$N(\alpha) = m + \left[\frac{1}{\pi^2}\log\left(\frac{1-\alpha}{\alpha}\right)\right]\log(m) + o(\log(m)).$$

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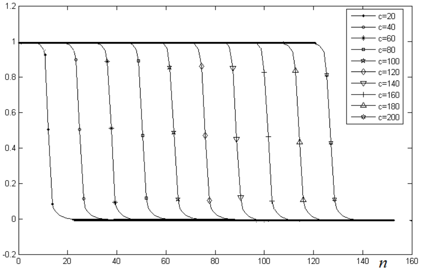

FIGURE : Graph of the $\lambda_j(\mathcal{Q}_m)$ for different values of $m = \frac{2c}{\pi}$

Non asymptotic results for $\lambda_j(\mathcal{Q}_m)$.

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Estimates obtained from min-max Principle.

Work in progress with A. Karoui and P. Jaming.

$$\lambda_n(\mathcal{Q}_m) = \min_{S_n} \max_{f \in S_n^\perp} \frac{\langle \mathcal{Q}_m f, f \rangle}{\|f\|^2_{L^2(I)}} = \min_{S_n} \max_{f \in S_n^\perp} \frac{\|\mathcal{F}_m f\|^2_{L^2(I)}}{\|f\|^2_{L^2(I)}},$$

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- By taking for S_n the $n+1-$dimensional space generated by Legendre polynomials of degree $\leq n$, we find the estimate, valid for $n \geq \pi em/4$,

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To be compared with asymptotic results of Widom and Fuchs.

Right plunge region and Remez Inequality.

Which $n+1-$dimensional space to be used for $m \approx n$?

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Take for S_n the space of functions that may be written as a translate of $\sum_{j=0}^{n} a_j \varphi((n+1)x - k)$, with φ compactly supported in I.

To bound below the ratio, have to consider the Remez type best constant $\Gamma_2(n, \varepsilon)$ such that

$$\int_J |P(e^{it})|^2 \, \mathrm{d}t \geq \Gamma(n, \varepsilon) \int_0^{2\pi} |P(e^{it})|^2 \, \mathrm{d}t,$$

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Gabriel Metsu


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The main theorem for random matrices.

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What is expected ?

$$A^*A = \frac{m}{n^2} \sum_{\ell=1}^{n} \left(e^{2i\pi m Z_\ell (Y_k - Y_j)}\right)_{j,k=1,\cdots n}.$$

Take the expectation in Z.

$$H := \mathbb{E}_Z(A^*A) = \frac{1}{n} \left(\frac{\sin(\pi m(Y_k - Y_j))}{\pi(Y_k - Y_j)}\right)_{j,k=1,\cdots n} := \frac{1}{n} \left(\kappa_m(Y_j, Y_k)\right)_{j,k=1,\cdots n}.$$

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$\kappa_m(x,y) = m\,\mathrm{sinc}(m(x-y))$ is a positive definite kernel.
Expected : the spectrum of A^*A is close to the spectrum of its expectation in Z, which is close to the spectrum of the integral operator $\mathcal{Q}_m$, that is, the operator with kernel κ_m with large probability.

McDiarmid 's Vectorial Concentration Inequality.

Theorem [Hayes]. Let Φ be a measurable function on $\mathbb{R}^n$ with values in a Hilbert space $\mathcal{H}$. Assume that

$$\|\Phi(z_1, \cdots, z_\ell, \cdots, z_n) - \Phi(z_1, \cdots, z'_\ell, \cdots, z_n)\|_{\mathcal{H}} \leq c$$

for each sequence $(z_j)_{j \neq \ell}, z_\ell, z'_\ell$. Then, if $Z_1, \cdots Z_n$ are independent,

$$\mathbb{P}\left(\|\Phi(Z_1, \cdots, Z_n) - \mathbb{E}\Phi(Z_1, \cdots, Z_n)\|_{\mathcal{H}} > \xi\right) \leq 2e^2 e^{-\frac{\xi^2}{2c^2 n}}.$$

For $n = 1$, called Mc Diarmid's Inequality, or Bounded Difference Inequality (Azuma-Hoeffding Inequality).

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Corollary. $\mathbb{P}_Z\left(\|A^*A - H\|_{\mathrm{HS}} > \xi \frac{m}{\sqrt{n}}\right) \leq 2e^2 e^{-\frac{\xi^2}{2}}$.

The first step.

Equivalent to say that, with a probability larger than $1 - 2e^2 e^{-\frac{\xi^2}{2}}$,

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Recall the notation : $\lambda(M) = \{\lambda_0(M), \cdots \lambda_{n-1}(M)\}$ is the spectrum of a positive definite matrix, in decreasing order.

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By Lindskii Inequality, this implies that

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We want the left hand side to be small compared to $\|H\|_{\mathrm{HS}}$.

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$$\mathbb{E}_Y(\|H\|_{\mathrm{HS}}^2) = m + O(\log m) + O\left(\frac{m^2}{n}\right).$$

The relative error will be small if m is large and m is small compared to n.

The second step.

More generally, let κ be a positive definite kernel on $(-1/2, +1/2)$ and assume that $\sup \kappa(x, x) \leq m$. Then, with probability larger than $1 - 2e^2 e^{-\frac{\xi^2}{2}}$, the spectrum of the random matrix

$$H_\kappa := n^{-1} \left(\kappa(Y_j, Y_k)\right)_{j,k=1,\cdots n}$$

satisfies the inequality

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Remark. Estimates for one eigenvalue are given by Shawe-Taylor, Cristianini and Kandola (see also Blanchard Bousquet and Wald). Such a matrix is called a kernel matrix or a kernel Gram matrix. For machine learning non linear principal component analysis (i. e. eigenvectors of such matrices).

Comparison with the integral operator.

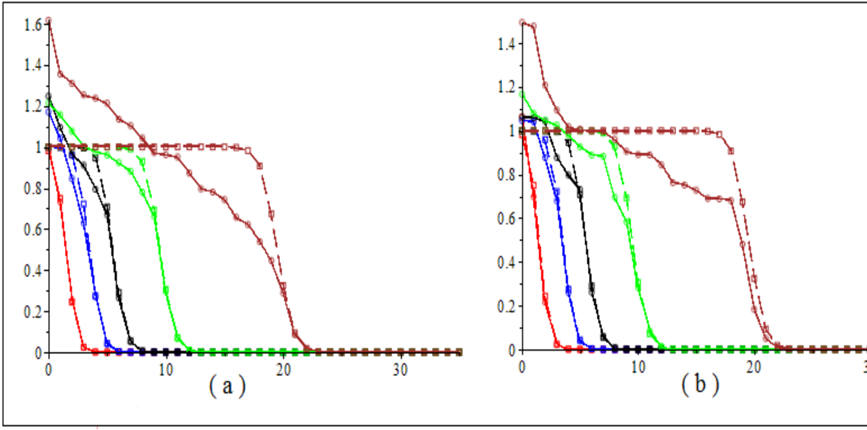

FIGURE : (a) Graphs of $\lambda(H)$ versus $\lambda(\mathcal{Q}_m)$ with $n = 200$ and for the various values of $m = 2, 4, 6, 10, 20$, (from the left to the right), (b) same as (a) with $\lambda(A^*A)$ instead of $\lambda(H)$.

Approximation by the integral operator.

We come back to

$$A^*A = \frac{m}{n^2}\sum_{\ell=1}^{n}\left(e^{2i\pi m Z_\ell (Y_k - Y_j)}\right)_{j,k=1,\cdots n}.$$

$$H := \mathbb{E}_Z(A^*A) = \frac{1}{n}\left(\frac{\sin(\pi m(Y_k - Y_j))}{\pi(Y_k - Y_j)}\right)_{j,k=1,\cdots n}.$$

Recall : want to prove that the spectrum of A^*A and H are close to the spectrum of $\mathcal{Q}_m$.
It remains to prove that the spectra of $\mathbb{E}(H)$ and $\mathcal{Q}_m$ are close.

A variant of Koltchinskii–Giné Theorem.

Theorem. *Let κ be a positive definite kernel on $(-1/2, +1/2)$ and T_κ be the integral operator with kernel κ and $\lambda(T_\kappa)$ its spectrum in decreasing order. Let*

$$H_\kappa := n^{-1} \left(\kappa(Y_j, Y_k)\right)_{j,k=1,\cdots n}.$$

Then, assuming that $m := \sup_y \kappa(y, y)$ is finite, one has the inequality

$$\mathbb{E}\left(\sum_{j=0}^{n-1} |\lambda_j(H_\kappa) - \lambda_j(T_\kappa)|^2\right) \leq \frac{2m^2}{n}.$$

One word on the proof of the theorem.

Let κ be a positive definite kernel on $(-1/2, +1/2)$, which is the kernel of the compact operator T. By Mercer Theorem it may be written as

$$\kappa(x, y) = \sum_j \lambda_j(T)\psi_j(x)\psi_j(y),$$

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Mercer's Formula is also used in machine learning to define a feature function, with values in a feature space. Namely the function $x \mapsto \Psi(x) = (\sqrt{\lambda_j(T)}\psi_j(x))_{j\geq 0}$. The feature space is the space ℓ^2 and

$$k(x, y) = \langle \Psi(x), \Psi(y) \rangle_{\ell^2}.$$

Conclusion for the spectra.

Putting together inequalities, , with probability larger than $1 - 2e^2 e^{-\frac{\xi^2}{2}}$,

$$\|\lambda(H) - \lambda(\mathcal{Q}_m)\|_{\ell^2} \leq \frac{(\xi + 1.5)m}{\sqrt{n}}.$$

Similarly, with probability larger than $1 - 2e^2 e^{-\frac{\xi^2}{2}}$,

$$\|\lambda(A^*A) - \lambda(\mathcal{Q}_m)\|_{\ell^2} \leq \frac{(2\xi + 1.5)m}{\sqrt{n}}.$$

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For m/n small, practical computations can be done on the integral operator.

Comparison with other results.

For a given eigenvalue,

$$|\lambda_j(H) - \lambda_j(\mathcal{Q}_m)| \leq \|\lambda(H) - \lambda(\mathcal{Q}_m)\|_{\ell^2} \leq \frac{(\xi + 1.5)m}{\sqrt{n}},$$

See the work of Shawe-Taylor, Cristianini and Kandola, and also Blanchard Bousquet and Wald... where such inequalities for one eigenvalue are developed for kernel matrices. But here all eigenvalues at the same time.

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Other results of these authors :

- $$\mathbb{E}(\sum_{j \geq s} \lambda_j(H)) \leq \sum_{j \geq s} \lambda_j(\mathcal{Q}_m).$$
- With large probability
$$|\sum_{j \geq s} \lambda_j(H) - \mathbb{E}(\sum_{j \geq s} \lambda_j(H))| \leq \frac{\xi m}{\sqrt{n}}.$$

Reconstruction error.

Because of the min-max principle

$$\sum_{j<s} \lambda_j(A)^2 = \sum_{j<s} \lambda_j(A^*A) = \sup \sum_{i=1}^{s} \langle A^*Av_i, v_i \rangle,$$

where the supremum is taken over all orthonormal system of s vectors.

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$$\mathbb{E}_Z(\sum_{j \geq s} \lambda_j(A)^2) \leq \sum_{j \geq s} \lambda_j(H).$$

$$\mathbb{E}(\sum_{j \geq s} \lambda_j(A)^2) \leq \sum_{j \geq s} \lambda_j(\mathcal{Q}_m).$$

Comparison with the integral operator.

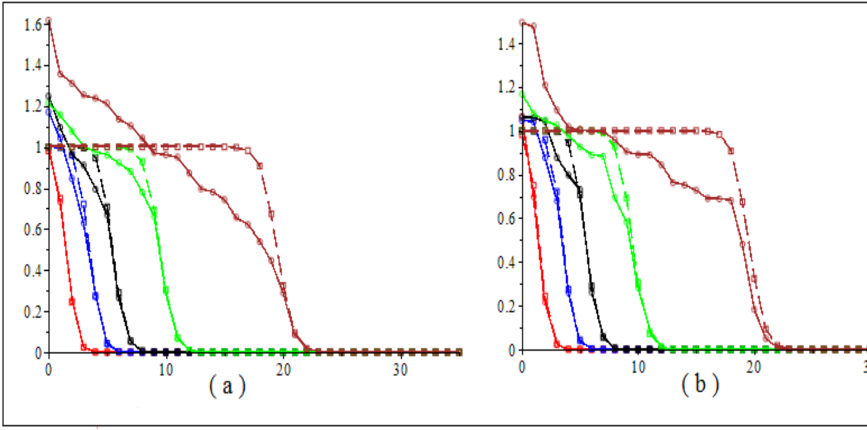

FIGURE : (a) Graphs of $\lambda(H)$ versus $\lambda(\mathcal{Q}_m)$ with $n = 200$ and for the various values of $m = 2, 4, 6, 10, 20$, (from the left to the right), (b) same as (a) with $\lambda(A^*A)$ instead of $\lambda(H)$.