Ramanujan-Fourier expansions of arithmetic functions of several variables

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6th Workshop on Fourier Analysis and Related Fields
Pécs, Hungary
24-31 August 2017

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$$= \frac{\pi^2}{6} \left(1 + \frac{(-1)^n}{2^2} + \frac{2\cos(2\pi n/3)}{3^2} + \frac{2\cos(\pi n/2)}{4^2} + \cdots \right),$$

absolutely convergent, where ζ is the Riemann zeta function.

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- Such expansions are called Ramanujan-Fourier (or Ramanujan) expansions of arithmetic functions.

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- The case of multiplicative functions was investigated by several authors ...

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$$c_{p^\nu}(n) = \begin{cases} p^\nu - p^{\nu-1}, & \text{if } p^\nu \mid n, \\ -p^{\nu-1}, & \text{if } p^\nu \nmid n, p^{\nu-1} \mid n, \\ 0, & \text{if } p^{\nu-1} \nmid n. \end{cases}$$

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- The set $\mathcal{E}_q$ of even functions $(\text{mod } q)$ forms an algebra with the usual linear operations.
- Moreover, $\mathcal{E}_q$ is a complex Hilbert space of finite dimension $\tau(q)$ under the inner product

$$\langle f, g \rangle = \frac{1}{q} \sum_{d|q} \varphi(d) f(r/d) \overline{g(q/d)},$$

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and $(c'_t)_{t|q}$, $c'_t(n) = \frac{1}{\sqrt{\varphi(t)}} c_t(n)$ is an orthonormal basis for $\mathcal{E}_r$.

Applications:

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deduced by E. Cohen (1966) and L. Tóth (2014), using different arguments.

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- Orthogonality relation I: for every $m, n \in \mathbb{N}$,

$$\frac{1}{[m,n]} \sum_{k=1}^{[m,n]} c_m(k)c_n(k) = \begin{cases} \varphi(n), & m = n, \\ 0, & \text{otherwise.} \end{cases}$$

- Consequence: Orthogonality relation II:

$$\sum_{n \le x} c_{q_1}(n) c_{q_2}(n) = \begin{cases} \varphi(q)x + O(1), & \text{if } q_1 = q_2 = q, \\ O(1), & \text{otherwise,} \end{cases}$$

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$$M(c_{q_1} c_{q_2}) = \begin{cases} \varphi(q), & \text{if } q_1 = q_2 = q, \\ 0, & \text{if } q_1 \neq q_2. \end{cases}$$

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where the Ramanujan-Fourier coefficients a_q are

$$a_q = \frac{1}{\varphi(q)} M(fc_q),$$

if the mean values $M(fc_q)$ exist for every $q \in \mathbb{N}$.

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- Theorem. There exist functions f such that the series $\sum_{q=1}^{\infty} a_q c_q(n)$ is divergent for every $q \in \mathbb{N}$.

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- The inverse of the constant 1 function under $*$ is μ_k, given by

$$\mu_k(n_1, \ldots, n_k) = \mu(n_1) \cdots \mu(n_k) \quad (n_1, \ldots, n_k \in \mathbb{N}),$$

where μ is the (classical) Möbius function.

- A function $f \in \mathcal{A}_k$ is said to be multiplicative if it is not identically zero and

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holds for any $m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbb{N}$ such that $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$.

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- Similar to the one dimensional case, the Dirichlet convolution preserves the multiplicativity of functions.

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- The proofs of the theorem and of its consequences are simplified, as well. They do not use mean value theorems, only properties of Ramanujan sums.
- Remark that, according to the generalized Wintner theorem, under conditions of the above theorem, the mean value

$$M(f) = \lim_{x_1,\ldots,x_k\to\infty} \frac{1}{x_1\cdots x_k} \sum_{n_1\le x_1,\ldots,n_k\le x_k} f(n_1,\ldots,n_k)$$

exists and $a_{1,\ldots,1} = M(f)$.

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Then for every $n_1,\ldots,n_k \in \mathbb{N}$ one has the absolutely convergent expansion of above with coefficients

$$a_{q_1,\ldots,q_k} = \prod_{p \in \mathbb{P}} \sum_{\nu_1 \ge \nu_p(q_1),\ldots,\nu_k \ge \nu_p(q_k)} \frac{(\mu_k * f)(p^{\nu_1},\ldots,p^{\nu_k})}{p^{\nu_1+\cdots+\nu_k}}.$$

Corollary

Let $g : \mathbb{N} \to \mathbb{C}$ be an arithmetic function and let $k \in \mathbb{N}$. Assume that

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Corollary

For every $n_1, \ldots, n_k \in \mathbb{N}$ the following series are absolutely convergent:

$$\frac{\sigma((n_1, \ldots, n_k))}{(n_1, \ldots, n_k)} = \zeta(k+1) \sum_{q_1, \ldots, q_k = 1}^{\infty} \frac{c_{q_1}(n_1) \cdots c_{q_k}(n_k)}{Q^{k+1}} \quad (k \geq 1),$$

$$\tau((n_1, \ldots, n_k)) = \zeta(k) \sum_{q_1, \ldots, q_k = 1}^{\infty} \frac{c_{q_1}(n_1) \cdots c_{q_k}(n_k)}{Q^{k}} \quad (k \geq 2).$$

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