Ramanujan-Fourier expansions of arithmetic functions of several variables

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Introduction

Ramanujan’s sums

Arithmetic functions of several variables

Results

The Ramanujan sums $c_q(n)$ are defined as the sum of $n$-th powers of the primitive $q$-th roots of unity ($q, n \in \mathbb{N}$), that is,

$$c_q(n) = \sum_{1 \leq k \leq q, (k, q) = 1} \exp(\frac{2\pi i k n}{q}).$$

S. Ramanujan (1918) derived pointwise convergent series representations of arithmetic functions with respect to these sums. For example, let $\sigma(n)$ denote the sum of divisors of $n$.

For every fixed $n \in \mathbb{N}$,

$$\sigma(n) = \zeta(2) \sum_{q=1}^{\infty} c_q(n) q^2 = \frac{\pi^2}{6} \left(1 + (-1)^{n/2} + 2 \cos(\frac{2\pi n}{3}) + 2 \cos(\frac{\pi n}{2}) + \cdots\right),$$

absolutely convergent, where $\zeta$ is the Riemann zeta function.
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$$= \frac{\pi^2}{6} \left(1 + \frac{(-1)^n}{2^2} + \frac{2 \cos(2\pi n/3)}{3^2} + \frac{2 \cos(\pi n/2)}{4^2} + \cdots\right),$$

absolutely convergent, where $\zeta$ is the Riemann zeta function.
This shows how the values of $\sigma(n)/n$ fluctuate harmonically about their mean value $\pi^2/6$.

The mean value of $f$ is $M(f) = \lim_{x \to \infty} \frac{1}{x} \sum_{n \leq x} f(n)$, if it exists.

Let $\tau(n)$ denote the number of divisors of $n$. Another result of Ramanujan:

$$\tau(n) = -\sum_{q=1}^{\infty} \log q \sigma_q(n),$$

convergent for any $n \in \mathbb{N}$. Note that the mean value $M(\tau)$ does not exist.

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Properties of Ramanujan sums:

\[ c_{q}(0) = \phi(q) \] is Euler's function,

\[ c_{q}(1) = \mu(q) \] is the Möbius function.

For every \( q, n \in \mathbb{N} \),

\[ c_{q}(n) = \sum_{d | (q, n)} d \mu\left(\frac{q}{d}\right), \]

hence \( c_{q}(n) \) is integer-valued.

For every fixed \( n \in \mathbb{N} \), the function \( q \mapsto c_{q}(n) \) is multiplicative, that is

\[ c_{q_{1}q_{2}}(n) = c_{q_{1}}(n)c_{q_{2}}(n), \]

provided that \((q_{1}, q_{2}) = 1\),

\[ c_{p^{\nu}}(n) = \begin{cases} p^{\nu} - p^{\nu-1}, & \text{if } p^{\nu} | n, \\ -p^{\nu} - 1, & \text{if } p^{\nu} \not| n, \\ p^{\nu} - 1 | n, \\ 0, & \text{if } p^{\nu-1} \not| n. \end{cases} \]
Properties of Ramanujan sums:

- \( c_{q}(0) = \varphi(q) \) is Euler's function,
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• A function $f : \mathbb{N} \to \mathbb{C}$ is called even (mod $q$) if $f(n) = f((n, q))$ for every $n$, that is the value $f(n)$ depends only on the gcd $(n, q)$.
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Moreover, $\mathcal{E}_q$ is a complex Hilbert space of finite dimension $\tau(q)$ under the inner product

$$\langle f, g \rangle = \frac{1}{q} \sum_{d \mid q} \varphi(d) f(r/d) \overline{g(q/d)},$$
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and \((c'_t)_{t | q}, c'_t(n) = \frac{1}{\sqrt{\varphi(t)}} c_t(n)\) is an orthonormal basis for \( \mathcal{E}_r \).
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Results

Consider the congruence

\[ x_1 + x_2 + \cdots + x_k \equiv n \pmod{q}. \]

The number of solutions \((\bmod q)\) such that \((x_i, q) = 1 \ (1 \leq i \leq k)\) is given by

\[ N(n, r, k) = \frac{1}{q} \sum_{d \mid q} c_q \left( \frac{q}{d} \right)^k c_d (n), \]

which goes back to the work of H. Rademacher and A. Brauer (1925), K. Ramanathan (1944).

The number \(N_k(n, q)\) of solutions of the congruence

\[ x_2 + \cdots + x_k \equiv n \pmod{q} \]

is, for \(k = 4 \) and \(q\) odd,

\[ N_k(n, r) = \frac{1}{2} \sum_{d \mid q} (d - 1) / 2 c_d (n), \]


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Applications:

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Hölder’s identity: for every $q, n \in \mathbb{N}$,

$$c_q(n) = \frac{\varphi(q)\mu(q/(n, q))}{\varphi(q/(n, q))}.$$


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Orthogonality relation I: for every \( m, n \in \mathbb{N} \),

\[
\frac{1}{[m, n]} \sum_{k=1}^{[m,n]} c_m(k)c_n(k) = \begin{cases} 
\varphi(n), & m = n, \\
0, & otherwise.
\end{cases}
\]
Consequence: Orthogonality relation II:

\[
\sum_{n \leq x} c_{q_1}(n)c_{q_2}(n) = \begin{cases} 
\varphi(q)x + O(1), & \text{if } q_1 = q_2 = q, \\
O(1), & \text{otherwise}, 
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M(c_{q_1}c_{q_2}) = \begin{cases} 
\varphi(q), & \text{if } q_1 = q_2 = q, \\
0, & \text{if } q_1 \neq q_2.
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This suggests to have expansions, convergent pointwise or in other sense, of functions $f$ of the form

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where the Ramanujan-Fourier coefficients $a_q$ are

$$a_q = \frac{1}{\varphi(q)} M(fc_q),$$

if the mean values $M(fc_q)$ exist for every $q \in \mathbb{N}$. 
Theorem. Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative such that $|f(n)| \leq 1$ $(n \in \mathbb{N})$ and $M(f) \neq 0$ exists. Then
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i) the Ramanujan coefficients $a_q$ exist for every $q \in \mathbb{N}$, $a_1 = M(f)$ and the map $q \mapsto a_q / a_1$ is multiplicative,

ii) one has the Parseval identity

$$\sum_{q=1}^{\infty} \varphi(q)|a_q|^2 = M(|f|^2),$$

The proof is very difficult, based on mean value theorems ...

Theorem. There exist functions $f$ such that the series

$$\sum_{q=1}^{\infty} a_q c_q(n)$$

is divergent for every $q \in \mathbb{N}$. 

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Theorem. Let \( f : \mathbb{N} \to \mathbb{C} \) be multiplicative such that \( |f(n)| \leq 1 \) \((n \in \mathbb{N})\) and \( M(f) \neq 0 \) exists. Then
i) the Ramanujan coefficients \( a_q \) exist for every \( q \in \mathbb{N} \), \( a_1 = M(f) \) and the map \( q \mapsto a_q / a_1 \) is multiplicative,
ii) one has the Parseval identity
\[
\sum_{q=1}^{\infty} \varphi(q) |a_q|^2 = M(|f|^2),
\]
iii) \( f(n) = \sum_{q=1}^{\infty} a_q c_q(n) \) is convergent for every \( n \in \mathbb{N} \).
Theorem. Let \( f : \mathbb{N} \to \mathbb{C} \) be multiplicative such that \( |f(n)| \leq 1 \) \((n \in \mathbb{N})\) and \( M(f) \neq 0 \) exists. Then

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The proof is very difficult, based on mean value theorems ...
Theorem. Let $f : \mathbb{N} \to \mathbb{C}$ be multiplicative such that $|f(n)| \leq 1$ ($n \in \mathbb{N}$) and $M(f) \neq 0$ exists. Then
i) the Ramanujan coefficients $a_q$ exist for every $q \in \mathbb{N}$, $a_1 = M(f)$ and the map $q \mapsto a_q/a_1$ is multiplicative,
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iii) $f(n) = \sum_{q=1}^{\infty} a_q c_q(n)$ is convergent for every $n \in \mathbb{N}$.

The proof is very difficult, based on mean value theorems ...

Theorem. There exist functions $f$ such that the series $\sum_{q=1}^{\infty} a_q c_q(n)$ is divergent for every $q \in \mathbb{N}$.
For a fixed \( k \in \mathbb{N} \) the set \( A_k \) of arithmetic functions \( f : \mathbb{N}^k \to \mathbb{C} \) of \( k \) variables is an integral domain with pointwise addition and the Dirichlet convolution defined by

\[
(f \ast g)(n_1, \ldots, n_k) = \sum_{d_1 | n_1, \ldots, d_k | n_k} f(d_1, \ldots, d_k) g(n_1/d_1, \ldots, n_k/d_k),
\]

the unity being the function \( \delta_k \), where

\[
\delta_k(n_1, \ldots, n_k) = \begin{cases} 1, & \text{if } n_1 = \cdots = n_k = 1, \\ 0, & \text{otherwise}. \end{cases}
\]

The inverse of the constant 1 function under \( \ast \) is \( \mu_k \), given by

\[
\mu_k(n_1, \ldots, n_k) = \mu(n_1) \cdots \mu(n_k) \quad (n_1, \ldots, n_k \in \mathbb{N})
\]

where \( \mu \) is the (classical) Möbius function.
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A function $f \in A_k$ is said to be multiplicative if it is not identically zero and $f(m_1, n_1, \ldots, m_k, n_k) = f(m_1, \ldots, m_k)f(n_1, \ldots, n_k)$ holds for any $m_1, \ldots, m_k, n_1, \ldots, n_k \in \mathbb{N}$ such that $(m_1 \cdots m_k, n_1 \cdots n_k) = 1$.

If $f$ is multiplicative, then it is determined by the values $f(p^{\nu_1}, \ldots, p^{\nu_k})$, where $p$ is prime and $\nu_1, \ldots, \nu_k \geq 0$.

More exactly, $f(1, \ldots, 1) = 1$ and for any $n_1, \ldots, n_k \in \mathbb{N}$, $f(n_1, \ldots, n_k) = \prod_{p \in \mathcal{P}} f(p^{\nu_p(n_1)}, \ldots, p^{\nu_p(n_k)})$. Similar to the one dimensional case, the Dirichlet convolution preserves the multiplicativity of functions.
A function \( f \in \mathcal{A}_k \) is said to be multiplicative if it is not identically zero and

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Similar to the one dimensional case, the Dirichlet convolution preserves the multiplicativity of functions.
Theorem (L. Tóth, Ramanujan J., 2017+)

Let $f: \mathbb{N}^k \to \mathbb{C}$ be an arithmetic function ($k \in \mathbb{N}$).

Assume that

$$\sum_{n_1, \ldots, n_k = 1}^\infty \omega(n_1) + \cdots + \omega(n_k) | (\mu_k^* f)(n_1, \ldots, n_k) | n_1 \cdots n_k < \infty.$$ 

Then for every $n_1, \ldots, n_k \in \mathbb{N},$

$$f(n_1, \ldots, n_k) = \sum_{q_1, \ldots, q_k = 1}^\infty a_{q_1, \ldots, q_k} c_{q_1}(n_1) \cdots c_{q_k}(n_k),$$

where

$$a_{q_1, \ldots, q_k} = \sum_{m_1, \ldots, m_k = 1}^\infty (\mu_k^* f)(m_1 q_1, \ldots, m_k q_k) m_1 q_1 \cdots m_k q_k,$$

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the series being absolutely convergent.
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In the one variable case this recovers a theorem H. Delange (Acta Arith., 1976).

For $k = 2$ we reobtain the recent result by N. Ushiroya (Hardy-Ramanujan J., 2016).

The proofs of the theorem and of its consequences are simplified, as well.

They do not use mean value theorems, only properties of Ramanujan sums.

Remark that, according to the generalized Wintner theorem, under conditions of the above theorem, the mean value $M(f) = \lim_{x_1,\ldots,x_k \to \infty} \sum_{n_1 \leq x_1,\ldots,n_k \leq x_k} f(n_1,\ldots,n_k)$ exists and $a_1,\ldots,a_k = M(f)$. 
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exists and \( a_1, \ldots, 1 = M(f) \).
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Results

Let \( f : \mathbb{N}^k \to \mathbb{C} \) be a multiplicative function (\( k \in \mathbb{N} \)). Assume that

\[
\sum_{p \in \mathbb{P}} \left| \left( \mu_k \ast f \right)(p^{\nu_1}, \ldots, p^{\nu_k}) \right| p^{\nu_1 + \cdots + \nu_k} < \infty.
\]

Then for every \( n_1, \ldots, n_k \in \mathbb{N} \) one has the absolutely convergent expansion of above with coefficients

\[
\prod_{p \in \mathbb{P}} \sum_{\nu_1 \geq \nu_p(\nu_1)} \cdots \sum_{\nu_k \geq \nu_p(\nu_k)} \left( \mu_k \ast f \right)(p^{\nu_1}, \ldots, p^{\nu_k}) p^{\nu_1 + \cdots + \nu_k}.
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Corollary

Let \( f : \mathbb{N}^k \to \mathbb{C} \) be a multiplicative function (\( k \in \mathbb{N} \)). Assume that

\[
\sum_{p \in \mathbb{P}} \sum_{\substack{\nu_1, \ldots, \nu_k = 0 \\ \nu_1 + \ldots + \nu_k \geq 1}} \frac{|(\mu_k \ast f)(p^{\nu_1}, \ldots, p^{\nu_k})|}{p^{\nu_1 + \ldots + \nu_k}} < \infty.
\]
Let $f : \mathbb{N}^k \to \mathbb{C}$ be a multiplicative function ($k \in \mathbb{N}$). Assume that

$$\sum_{p \in \mathbb{P}} \sum_{\nu_1, \ldots, \nu_k = 0}^{\infty} \frac{|(\mu_k \ast f)(p^{\nu_1}, \ldots, p^{\nu_k})|}{p^{\nu_1 + \ldots + \nu_k}} < \infty.$$ 

Then for every $n_1, \ldots, n_k \in \mathbb{N}$ one has the absolutely convergent expansion of above with coefficients

$$a_{q_1, \ldots, q_k} = \prod_{p \in \mathbb{P}} \sum_{\nu_1 \geq \nu_p(q_1), \ldots, \nu_k \geq \nu_p(q_k)} \frac{(\mu_k \ast f)(p^{\nu_1}, \ldots, p^{\nu_k})}{p^{\nu_1 + \ldots + \nu_k}}.$$
Let $g : \mathbb{N} \rightarrow \mathbb{C}$ be an arithmetic function and let $k \in \mathbb{N}$. Assume that
\[ \sum_{n=1}^{\infty} 2^k \omega(n) | (\mu \ast g)(n) | n^k < \infty. \]
Then for every $n_1, \ldots, n_k \in \mathbb{N}$,
\[ g((n_1, \ldots, n_k)) = \sum_{q_1, \ldots, q_k=1}^{\infty} a_{q_1, \ldots, q_k} c_{q_1}(n_1) \cdots c_{q_k}(n_k), \]
with
\[ a_{q_1, \ldots, q_k} = 1_{Q_k} \sum_{m=1}^{\infty} (\mu \ast g)(mq^k) m^k, \]
where $Q_k = [q_1, \ldots, q_k]$. 

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Corollary

Let $g : \mathbb{N} \to \mathbb{C}$ be an arithmetic function and let $k \in \mathbb{N}$. Assume that

$$\sum_{n=1}^{\infty} 2^k \omega(n) \frac{|(\mu * g)(n)|}{n^k} < \infty.$$
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Corollary

Let \( g : \mathbb{N} \rightarrow \mathbb{C} \) be an arithmetic function and let \( k \in \mathbb{N} \). Assume that

\[
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\]

Then for every \( n_1, \ldots, n_k \in \mathbb{N} \),

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\[
a_{q_1, \ldots, q_k} = \frac{1}{Q^k} \sum_{m=1}^{\infty} \frac{\left( \mu \ast g \right)(mQ)}{m^k},
\]

where \( Q = [q_1, \ldots, q_k] \).
For every $n_1, \ldots, n_k \in \mathbb{N}$ the following series are absolutely convergent:

$$\sigma((n_1, \ldots, n_k)) (n_1, \ldots, n_k) = \zeta(k+1) \infty \sum_{q_1, \ldots, q_k=1} c_{q_1}(n_1) \cdots c_{q_k}(n_k) Q_{k+1}(k \geq 1),$$

$$\tau((n_1, \ldots, n_k)) = \zeta(k) \infty \sum_{q_1, \ldots, q_k=1} c_{q_1}(n_1) \cdots c_{q_k}(n_k) Q_k(k \geq 2).$$
Corollary

For every $n_1, \ldots, n_k \in \mathbb{N}$ the following series are absolutely convergent:

$$\sigma\left(\left(\frac{n_1, \ldots, n_k}{n_1, \ldots, n_k}\right)\right) = \zeta(k + 1) \sum_{q_1, \ldots, q_k=1}^{\infty} \frac{c_{q_1}(n_1) \cdots c_{q_k}(n_k)}{Q^{k+1}} \quad (k \geq 1),$$

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For every $n_1, \ldots, n_k \in \mathbb{N}$, the following series are absolutely convergent:

$$
\varphi((n_1, \ldots, n_k)) (n_1, \ldots, n_k) = 1 \quad \zeta(k+1) \quad \sum_{q_1, \ldots, q_k=1}^{\infty} \mu(Q) c_{q_1}(n_1) \cdots c_{q_k}(n_k) \varphi_{k+1}(Q) \quad (k \geq 1),
$$
Corollary

For every $n_1, \ldots, n_k \in \mathbb{N}$ the following series are absolutely convergent:

$$\frac{\varphi((n_1, \ldots, n_k))}{(n_1, \ldots, n_k)} = \frac{1}{\zeta(k + 1)} \sum_{q_1, \ldots, q_k=1}^{\infty} \frac{\mu(Q)c_{q_1}(n_1) \cdots c_{q_k}(n_k)}{\varphi_{k+1}(Q)} \quad (k \geq 1),$$
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$d$ is called a unitary divisor (or block divisor) of $n$ if $d \mid n$ and $(d, n/d) = 1$. Notation: $d \mathrel{||} n$.

Let $\sigma^* (n)$ denote the sum of unitary divisors of $n$.

Theory of functions defined by unitary divisors ...

The unitary Ramanujan sums $c^* q(n)$ were defined by E. Cohen (1960):

$$c^* q(n) = \sum_{1 \leq k \leq q} \left( (k, q) \ast = 1 \right) \exp \left( \frac{2\pi i k n}{q} \right) \quad (q, n \in \mathbb{N}),$$

where $(k, q) \ast = \max \{d : d \mid k, d \mathrel{||} q\}$.

Similar properties:

$$c^* q(n) = \sum d \mathrel{||} (n, q) \ast d \mu^* \left( \frac{q}{d} \right) \quad (q, n \in \mathbb{N})$$

However, the orthogonality properties are not valid.

It is possible to deduce similar results with respect to the sums $c^* q(n)$.
\( d \) is called a unitary divisor (or block divisor) of \( n \) if \( d \mid n \) and \((d, n/d) = 1\). Notation: \( d \parallel n \).
- $d$ is called a unitary divisor (or block divisor) of $n$ if $d \mid n$ and $(d, n/d) = 1$. Notation: $d \parallel n$.
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c_q^*(n) = \sum_{1 \leq k \leq q} \exp(2\pi i kn/q) \quad (q, n \in \mathbb{N}),
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    \[ c_q^*(n) = \sum_{1 \leq k \leq q \atop (k, q)_* = 1} \exp(2\pi i kn/q) \quad (q, n \in \mathbb{N}), \]
    where $(k, q)_* = \max\{d : d \mid k, d \| q\}$.
\end{itemize}
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c_q^*(n) = \sum_{1 \leq k \leq q, (k, q)_* = 1} \exp(2\pi i kn/q) \quad (q, n \in \mathbb{N}),
\]
  where $(k, q)_* = \max\{d : d \mid k, d \mid\mid q\}$.
- Similar properties:
  \[
c_q^*(n) = \sum_{d \mid\mid(n, q)_*} d\mu^*(q/d) \quad (q, n \in \mathbb{N}),
\]
- $d$ is called a unitary divisor (or block divisor) of $n$ if $d \mid n$ and $(d, n/d) = 1$. Notation: $d \| n$.
- Let $\sigma^*(n)$ denote the sum of unitary divisors of $n$.
- Theory of functions defined by unitary divisors ...
- The unitary Ramanujan sums $c^*_q(n)$ were defined by E. Cohen (1960):

$$c^*_q(n) = \sum_{1 \leq k \leq q, \ (k, q)_* = 1} \exp(2\pi i kn/q) \quad (q, n \in \mathbb{N}),$$

where $(k, q)_* = \max\{d : d \mid k, d \| q\}$.
- Similar properties:

$$c^*_q(n) = \sum_{d \| (n, q)_*} d\mu^*(q/d) \quad (q, n \in \mathbb{N}),$$

- The orthogonality properties are not valid.
• \(d\) is called a unitary divisor (or block divisor) of \(n\) if \(d \mid n\) and \((d, n/d) = 1\). Notation: \(d \parallel n\).
• Let \(\sigma^*(n)\) denote the sum of unitary divisors of \(n\).
• Theory of functions defined by unitary divisors ...
• The unitary Ramanujan sums \(c_q^*(n)\) were defined by E. Cohen (1960):

\[
c_q^*(n) = \sum_{1 \leq k \leq q} \exp\left(2\pi i \frac{kn}{q}\right) \quad (q, n \in \mathbb{N}),
\]

where \((k, q)_* = \max\{d : d \mid k, d \parallel q\}\). 
• Similar properties:

\[
c_q^*(n) = \sum_{d \parallel (n, q)_*} d\mu^*(q/d) \quad (q, n \in \mathbb{N}),
\]

• The orthogonality properties are not valid. However, it is possible to deduce similar results with respect to the sums \(c_q^*(n)\).
Theorem (L. Tóth, Ramanujan J., 2017+)

Let \( f : \mathbb{N}^k \rightarrow \mathbb{C} \) be an arithmetic function (\( k \in \mathbb{N} \)). Assume that
\[
\sum_{n_1, \ldots, n_k = 1} \omega(n_1) + \cdots + \omega(n_k) | (\mu_k \ast f)(n_1, \ldots, n_k) | n_1 \cdots n_k < \infty.
\]
Then for every \( n_1, \ldots, n_k \in \mathbb{N} \),
\[
f(n_1, \ldots, n_k) = \sum_{q_1, \ldots, q_k = 1} a \ast q_1 \cdots c \ast q_k \omega(q_1(n_1)) \cdots \omega(q_k(n_k)),
\]
\( a \ast q_1 \cdots c \ast q_k\) is
\[
\sum_{m_1, \ldots, m_k = 1} (m_1, q_1) = \cdots = (m_k, q_k) | (\mu_k \ast f)(m_1 q_1, \ldots, m_k q_k) | m_1 q_1 \cdots m_k q_k,
\]
the series being absolutely convergent.
### Theorem (L. Tóth, Ramanujan J., 2017+)

Let $f : \mathbb{N}^k \to \mathbb{C}$ be an arithmetic function ($k \in \mathbb{N}$).
Theorem (L. Tóth, Ramanujan J., 2017+)

Let $f : \mathbb{N}^k \rightarrow \mathbb{C}$ be an arithmetic function ($k \in \mathbb{N}$). Assume that

$$\sum_{n_1, \ldots, n_k = 1}^{\infty} 2^{\omega(n_1) + \cdots + \omega(n_k)} \left| \left( \mu_k \ast f \right)(n_1, \ldots, n_k) \right| \frac{\mu_k \ast f(n_1, \ldots, n_k)}{n_1 \cdots n_k} < \infty.$$
**Theorem (L. Tóth, Ramanujan J., 2017+)**

Let $f : \mathbb{N}^k \to \mathbb{C}$ be an arithmetic function ($k \in \mathbb{N}$). Assume that

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Then for every $n_1, \ldots, n_k \in \mathbb{N}$,

$$f(n_1, \ldots, n_k) = \sum_{q_1, \ldots, q_k = 1}^{\infty} a^*_{q_1, \ldots, q_k} c^*_{q_1}(n_1) \cdots c^*_{q_k}(n_k),$$
Theorem (L. Tóth, Ramanujan J., 2017+)

Let \( f : \mathbb{N}^k \to \mathbb{C} \) be an arithmetic function \((k \in \mathbb{N})\). Assume that

\[
\sum_{n_1, \ldots, n_k = 1}^{\infty} 2^{\sum_{j=1}^{k} \omega(n_j)} \frac{|(\mu_k \ast f)(n_1, \ldots, n_k)|}{n_1 \cdots n_k} < \infty.
\]

Then for every \( n_1, \ldots, n_k \in \mathbb{N} \),

\[
f(n_1, \ldots, n_k) = \sum_{q_1, \ldots, q_k = 1}^{\infty} a_{q_1,\ldots,q_k}^* c_{q_1}^*(n_1) \cdots c_{q_k}^*(n_k),
\]

where

\[
a_{q_1,\ldots,q_k}^* = \sum_{m_1, \ldots, m_k = 1}^{\infty} \frac{(\mu_k \ast f)(m_1 q_1, \ldots, m_k q_k)}{m_1 q_1 \cdots m_k q_k},
\]

the series being absolutely convergent.
For every $n_1, \ldots, n_k \in \mathbb{N}$ the following series are absolutely convergent:

$$\sigma((n_1, \ldots, n_k)) = \zeta(k+1) \sum_{q_1, \ldots, q_k=1}^{\infty} \phi_k+1(Q) c^* q_1(n_1) \cdots c^* q_k(n_k) Q^{2(k+1)}$$

$$\tau((n_1, \ldots, n_k)) = \zeta(k) \sum_{q_1, \ldots, q_k=1}^{\infty} \phi_k(Q) c^* q_1(n_1) \cdots c^* q_k(n_k) Q^{2k}$$
Corollary

For every $n_1, \ldots, n_k \in \mathbb{N}$ the following series are absolutely convergent:

$$\frac{\sigma((n_1, \ldots, n_k))}{(n_1, \ldots, n_k)} = \zeta(k + 1) \sum_{q_1, \ldots, q_k=1}^{\infty} \frac{\varphi_{k+1}(Q)c_{q_1}^*(n_1) \cdots c_{q_k}^*(n_k)}{Q^{2(k+1)}} \quad (k \geq 1),$$

$$\tau((n_1, \ldots, n_k)) = \zeta(k) \sum_{q_1, \ldots, q_k=1}^{\infty} \frac{\varphi_k(Q)c_{q_1}^*(n_1) \cdots c_{q_k}^*(n_k)}{Q^{2k}} \quad (k \geq 2).$$