The boundedness of the L^1-norm of Walsh-Fejér kernels

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Trigonometric system

The L^1-norm of kernels corresponding to operators related to orthonormal systems plays an important role in the convergence of orthogonal series.

Dirichlet kernels:

$$D_n(x) = \frac{1}{2\pi} + \frac{1}{\pi}\sum_{k=1}^{n} \cos k(x) = \frac{1}{\pi} \cdot \frac{sin(n+\frac{1}{2})(x)}{2\sin\frac{1}{2}(x)}$$

Lebesgue constants:

$$L_n = \int_0^{2\pi} |D_n(x)|\, dx = \frac{2}{\pi}\int_0^{\frac{\pi}{2}} \left|\frac{sin(2n+1)t}{\sin t}\right| dt$$

Trigonometric system

Fejér (1910)

$$L_n = \frac{4}{\pi^2} \log n + a_0 + \varepsilon_n \qquad (\varepsilon_n \to 0).$$

Thus $L_n = O(\log n)$.

Gábor Szegő (1921)

$$L_n = \frac{16}{\pi^2} \sum_{k=1}^{\infty} \frac{1}{4k^2 - 1} \left(1 + \frac{1}{3} + \frac{1}{5} + \cdots + \frac{1}{2k(2n+1) - 1}\right)$$

Fejér kernels:

$$K_n f(x) := \frac{D_0 f(x) + D_1 f(x) + \cdots + D_n f(x)}{n+1}$$

$$= \frac{1}{2(n+1)\pi} \left(\frac{\sin \frac{(n+1)(x)}{2}}{\sin \frac{x}{2}} \right)^2 \geq 0,$$

However

$$\int_0^{2\pi} |K_n(t,x)| \, dt = 1$$

The Walsh-Paley system

The binary expansion $n = (n_0, n_1, \dots)$ of the number $n \in \mathbf{N}$:

$$n = \sum_{k=0}^{\infty} n_k 2^k, \qquad (n_k = 0 \text{ or } n_k = 1)$$

The binary expansion $x = (x_0, x_1, \dots)$ of the number $x \in [0, 1[$:

$$x = \sum_{k=0}^{\infty} \frac{x_k}{2^{k+1}}, \qquad (x_k = 0 \text{ or } x_k = 1)$$

The Rademacher system:

$$r_k(x) := (-1)^{x_k},$$

where $k \in \mathbf{N}$ and $x \in [0, 1[$.

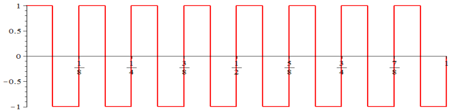

Figure: Rademacher function r_3

The Rademacher system is orthonormal, but not complete.

The Walsh-Paley system

Walsh functions:

The finite product of Rademacher functions.

The Walsh-Paley system:

$$\omega_n(x) := \prod_{k=0}^{\infty} r_k^{n_k}(x) = (-1)^{\sum_{k=0}^{\infty} x_k n_k} \qquad (x \in [0,1[, n \in \mathbf{N}).$$

The Walsh-Paley system is an orthonormal and complete system defined on the interval $[0, 1[$.

The Walsh-Paley system

Walsh-Dirichlet kernels:

$$D_n(x) := \sum_{k=0}^{n-1} \omega_k(x) \quad (x \in [0,1[)$$

Dyadic intervals:

$$I_k(i) := \left[\frac{i}{2^k}, \frac{i+1}{2^k}\right[\quad (i = 0, \ldots, 2^k - 1), \qquad I_k := I_k(0)$$

Paley's lemma:

$$D_{2^k}(x) = \begin{cases} 2^k, & x \in I_k, \\ 0, & x \in [0,1[\setminus I_k. \end{cases}$$

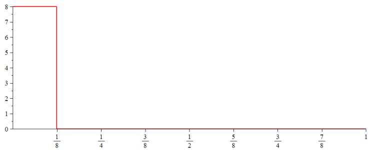

Figure: Dirichlet kernel D_8

Walsh-Lebesgue constants:

$$L_n := \int_0^1 |D_n(x)|\, dx \qquad (n \in \mathbf{N}).$$

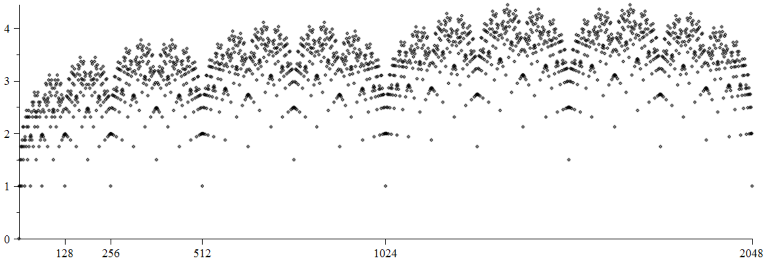

Figure: Walsh-Lebesgue constants

Walsh-Lebesgue constants

Fine (1949)

Iteration for Walsh-Lebesgue constants: If $n = 2^k + m$ where $0 \leq m < 2^k$, then

$$L_n = 1 + L_m - \frac{m}{2^k}.$$

Properties (Fine)

- $L_n = O(\log n)$
- $L_{2n} = L_n$
- $L_{2n+1} = \dfrac{1 + L_n + L_{n+1}}{2}$
- $\frac{1}{n}\sum_{k=1}^{n} L_k \geq C \log n$

The L^1-norm of Walsh-Fejér kernels

Walsh-Fejér kernels: If $x \in [0, 1[$

$$K_n(x) := \frac{1}{n}\sum_{k=1}^{n} D_k(x), \qquad \|K_n\|_1 := \int_0^1 |K_n(x)|\,dx$$

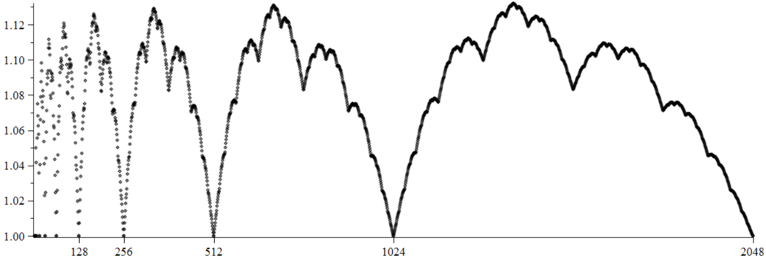

Figure: The L^1-norm of Walsh-Fejér kernels

The property $\|K_n\|_1 \leq 2$ holds for all $n \in \mathbf{P}$.

Yano (1951)

$$K_{2^k}(x) = \begin{cases} \frac{2^k+1}{2}, & x \in I_k, \\ 2^{j-1}, & x \in I_k(2^{k-j-1}),\ j = 0, 1, \ldots, k-1 \\ 0, & \text{elsewhere in } [0, 1[. \end{cases}$$

Note that if $x \in I_k(2^{k-j-1})$, then $x = (\underbrace{0, \ldots, 0}_{j}, 1, \underbrace{0, \ldots, 0}_{k-j-1}, x_k, x_{k+1}, \ldots)$

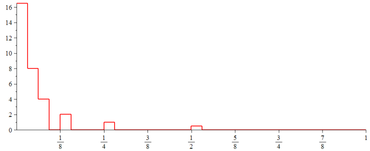

Figure: K_{32} Walsh-Fejér kernel K_{32}

Iteration for the L^1-norm of Walsh-Fejér kernels

Iteration for the L^1-norm of Walsh-Fejér kernels: It was unknown until now.

The idea: Let $n < 2^{k+1}$. Let us look at the values of $D_n(x)$ when $K_{2^k}(x)$ is positive.

Lemma

Let $k, n \in \mathbf{N}$ *such that* $n < 2^{k+1}$.

- *If* $x \in I_{k+1}$ *then* $D_n(x) = n$.
- *If* $x \in I_k \setminus I_{k+1}$ *then*

$$D_n(x) = \begin{cases} n, & n < 2^k, \\ 2^{k+1} - n, & 2^k \leq n < 2^{k+1}. \end{cases}$$

Lemma (continue)

- *Suppose* $x \in I_k(2^{k-j-1})$ *for some* $j = 0, 1, \dots, k-1$ *and let* p *and* n' *be non-negative integers such that* $n = p2^{j+1} + n'$ *holds, where* $0 \le n' < 2^{j+1}$. *If* $n < 2^k$ *or also, if* $n \ge 2^k$ *but* x *is in the first half of the interval* $I_k(2^{k-j-1})$ *then*

$$D_n(x) = \begin{cases} n', & n' < 2^j, \\ 2^{j+1} - n', & 2^j \le n' < 2^{j+1}. \end{cases}$$

Moreover, if $n \ge 2^k$ *and* x *is in the second half of the interval* $I_k(2^{k-j-1})$ *then*

$$D_n(x) = \begin{cases} -n', & n' < 2^j, \\ n' - 2^{j+1}, & 2^j \le n' < 2^{j+1}. \end{cases}$$

Iteration for the L^1-norm of Walsh-Fejér kernels

Corollary

If $n < 2^{k+1}$, then the value of $K_n(x)$ is nonnegative if the value of $K_{2^k}(x)$ is positive.

Lemma

Let $k, n \in \mathbf{N}$ such that $n < 2^{k+1}$.

- *If $x \in I_{k+1}$ then*

$$nK_n(x) = \frac{n(n+1)}{2}.$$

- *If $x \in I_k \setminus I_{k+1}$ then*

$$nK_n(x) = \begin{cases} \frac{n(n+1)}{2}, & n < 2^k, \\ \frac{n(n+1)}{2} - (n-2^k)(n-2^k+1), & 2^k \leq n < 2^{k+1}. \end{cases}$$

Lemma (continue)

- *Suppose* $x \in I_k(2^{k-j-1})$ *for some* $j = 0, 1, \ldots, k-1$ *and let* p *and* n' *be non-negative integers such that* $n = p2^{j+1} + n'$ *holds, where* $0 \leq n' < 2^{j+1}$. *If* $n < 2^k$ *or also, if* $n \geq 2^k$ *but* x *is in the first half of the interval* $I_k(2^{k-j-1})$ *then*

$$nK_n(x) = \begin{cases} p4^j + \frac{n'(n'+1)}{2}, & n' < 2^j, \\ p4^j + \frac{n'(n'+1)}{2} - (n' - 2^j)(n' - 2^j + 1), & 2^j \leq n' < 2^{j+1}. \end{cases}$$

Moreover, if $n \geq 2^k$ *and* x *is in the second half of the interval* $I_k(2^{k-j-1})$ *then*

$$nK_n(x) = \begin{cases} 2^{k+j} - p4^j - \frac{n'(n'+1)}{2}, & n' < 2^j, \\ 2^{k+j} - p4^j - \frac{n'(n'+1)}{2} + (n' - 2^j)(n' - 2^j + 1), & 2^j \leq n' < \end{cases}$$

Iteration for the L^1-norm of Walsh-Fejér kernels

New notation:

$$n'_j := \sum_{i=0}^{j} n_i 2^i = n \bmod 2^{j+1},$$

where j is a nonnegative integer.

Theorem

Let $k \in \mathbf{N}$ and n be a positive integer such that $2^k \leq n < 2^{k+1}$. Then, the following iteration is valid:

$$n\|K_n\|_1 = n + n'_{k-1}\|K_{n'_{k-1}}\|_1 - \frac{\Gamma_n}{2^{k+1}},$$

where

$$\Gamma_n := 2n'_{k-1}(n'_{k-1}+1) + \sum_{j=1}^{k-1}\left(n_j 2^j(2^j-1) + (1-2n_j)n'_{j-1}(n'_{j-1}+1)\right).$$

Iteration for the L^1-norm of Walsh-Fejér kernels

The essence of the proof:

$$n\|K_n\|_1 = \int_A |nK_n(x)|\, dx + \int_{\overline{A}} |nK_n(x)|\, dx := \mathcal{I}_1 + \mathcal{I}_2,$$

where

$$A = \bigcup_{j=0}^{k-1} I_k(2^{k-j-1}) \cup I_k$$

and use the iteration $nK_n = 2^k K_{2^k} + mD_{2^k} + r_k mK_m$.

$$\mathcal{I}_1 = \int_A 2^k K_{2^k}(x) + mD_{2^k}(x)\, dx + \int_A r_k(x) mK_m(x)\, dx = 2^k + m + 0 = n$$

$$\mathcal{I}_2 = \int_{\overline{A}} |0 + r_k(x) mK_m(x)|\, dx = \int_{\overline{A}} |mK_m(x)|\, dx = \|mK_m\|_1 - \int_A mK_m(x)\, dx$$

Iteration for the L^1-norm of Walsh-Fejér kernels

New notation: Let $n = 2^{k_0} + 2^{k_1} + \cdots + 2^{k_s}$, where $k_0 < k_1 < \cdots < k_s$. Denote

$$n'_{k_i} = \sum_{r=0}^{i} 2^{k_r}$$

where $i = 0, 1, \ldots, s$.

Theorem

Suppose that $n = 2^{k_0} + 2^{k_1} + \cdots + 2^{k_s}$, where $k_0 < k_1 < \cdots < k_s$ are nonnegative integers. Then

$$n\|K_n\|_1 = n + n'_{k_{s-1}} \|K_{n'_{k_{s-1}}}\|_1 - \frac{\Gamma_n}{2^{k_s+1}},$$

where

$$\Gamma_n = n'_{k_{s-1}}(3n'_{k_{s-1}} + 2) + \sum_{i=0}^{s-1} \left(4^{k_i} + (k_{i+1} - k_i - 2)n'_{k_i}(n'_{k_i} + 1)\right).$$

Properties of the L^1-norm of Walsh-Fejér kernels

Let $n = 2^{k_0} + 2^{k_1} + \cdots + 2^{k_s}$ and denote

$$\gamma_n := 2n'_{k_{s-1}} + \sum_{i=0}^{s-1} (k_{i+1} - k_i - 2) n'_{k_i} \qquad (n \in \mathbf{P})$$

Lemma

$\gamma_n \geq 0$ *and* $\gamma_n = 0$ *if and only if n is a power of 2.*

Theorem

$$\|K_{2n}\|_1 - \|K_n\|_1 = \frac{1}{4n} \left(\frac{\gamma_{n'_{k_0}}}{2^{k_0}} + \frac{\gamma_{n'_{k_1}}}{2^{k_1}} + \cdots + \frac{\gamma_{n'_{k_s}}}{2^{k_s}} \right) \geq 0$$

and the equality holds if and only if n is a power of 2.

Theorem

Let $k \in \mathbf{N}$. If $n, m \geq 2^k$ and $n + m = 3 \cdot 2^k - 1$, then

$$n\|K_n\|_1 - m\|K_m\|_1 = n - m.$$

Properties of the L^1-norm of Walsh-Fejér kernels

For a fixed positive integer k denote by $n^*(k)$ the index less than 2^{k+1} such that

$$\|K_{n^*(k)}\|_1 = \max\{\|K_n\|_1 : 1 \leq n < 2^{k+1}\}.$$

Theorem

Let $k \in \mathbf{P}$. Then

1. *If k is even then $n^*(k) = 1 + 2^2 + 2^4 + \cdots + 2^k$ and*

$$\|K_{n^*(k)}\|_1 = \frac{17}{15} - \frac{s+1}{4^{s+1}-1} + \frac{1}{5 \cdot 4^s},$$

   where $s = \frac{k}{2}$.

2. *If k is odd then $n^*(k) = 2^1 + 2^3 + \cdots + 2^k$ and*

$$\|K_{n^*(k)}\|_1 = \frac{17}{15} - \frac{1}{2}\frac{s+1}{4^{s+1}-1} + \frac{1}{30 \cdot 4^s},$$

   where $s = \frac{k-1}{2}$.

Properties of the L^1-norm of Walsh-Fejér kernels

Theorem

$$\sup\{\|K_n\|_1 : n \in \mathbf{P}\} = \frac{17}{15}$$

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Abstract

In this paper we establish an iteration for the L^1-norm of Walsh–Fejér kernels with the assumption that the Walsh functions are ordered in Paley's sense. We use this iteration to prove some properties of this sequence, including that its supremum is exactly equal to $\frac{17}{15}$.

Keywords

Fourier analysis; Walsh–Paley system; Walsh–Fejér kernels; Lebesgue constants

Thank you for your attention.