**Spectral synthesis holds on the reals.** In other words: given any continuous complex valued function $f$ on the reals it is the uniform limit on compact sets of linear combinations of exponential monomials of the form $x \mapsto x^n e^{\lambda x}$ ($n$ is a natural number, $\lambda$ is a complex number) such that all these exponential monomials belong to the smallest linear space including all translates of $f$ and being closed with respect to uniform convergence on compact sets.
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Laurent Schwartz, 1947

Laurent Schwartz

With his butterflies
No direct extension of Schwartz’s result to $\mathbb{R}^n$ is possible:

**Spectral synthesis fails to hold in $\mathbb{R}^n$ for $n \geq 2$**

(Dmitrii I. Gurevich, 1975) For each natural number $n \geq 2$ there exist compactly supported measures $\mu, \nu$ such that the exponential monomial solutions of the system of functional equations

$$\mu * f = 0, \quad \nu * f = 0$$

do not span a dense subspace in the solution space of this system.
Counterexamples

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**Spectral analysis fails to hold in $\mathbb{R}^n$ for $n \geq 2$**

(Dmitrii I. Gurevich, 1975) For each natural number $n \geq 2$ there exist compactly supported measures $\mu_1, \mu_2, \ldots, \mu_6$ such that the system

$$\mu_k \ast f = 0, \quad k = 1, 2, \ldots, 6$$

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$G$: locally compact Abelian group, $C(G)$: locally convex topological vector space of all continuous complex valued functions on $G$. 
Notation and terminology

$G$: locally compact Abelian group, $C(G)$: locally convex topological vector space of all continuous complex valued functions on $G$, topology: compact convergence

$\mathcal{M}_c(G)$: measure algebra
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Convolution:

\[\mu \ast \nu(f)(x) = \int_G \mu(\tau_y f)(x) \, d\nu(y),\]

where \(\tau_y f(x) = f(x - y)\).
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Spectral analysis and synthesis

**Spectral analysis for a variety:** every nonzero subvariety has a nonzero finite dimensional subvariety
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Spectral analysis and synthesis

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Basic function classes

Exponential:

Let $G$ be a locally compact Abelian group and $f: G \rightarrow \mathbb{C}$ a continuous function. Then the following conditions are equivalent.

1. $f$ is an exponential.
2. $\tau_p f$ is one dimensional and $f(0) = 1$.
3. $f$ is a normalized common eigenfunction of all translation operators.
4. $f$ is a normalized common eigenfunction of all convolution operators.

László Székelyhidi
Spectral Synthesis on Affine Groups
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Basic function classes

**Exponential monomial:** We let for each exponential $m$:

$$\Delta_{m;y} = \delta_{-y} - m(y)\delta_0,$$

the *modified difference* corresponding to $m$ with increment $y$. Higher order differences:

$$\Delta_{m;y_1,y_2,\ldots,y_{n+1}} = \prod_{k=1}^{n+1} \Delta_{m;y_k}.$$
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The continuous function $f : G \to \mathbb{C}$ is called an *exponential monomial* if $\tau(f)$ is finite dimensional and there exists an exponential $m$ and a natural number $n$ such that

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**Exponential polynomial:**

*linear combination of exponential monomials*

**Theorem**

Let \( G \) be an Abelian group. A variety on \( C_G \) is finite dimensional if and only if it is spanned by exponential monomials.
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**Exponential polynomial:** linear combination of exponential monomials

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**Exponential polynomial:** linear combination of exponential monomials

**Theorem**

*Let $G$ be an Abelian group. A variety on $\mathbb{C}G$ is finite dimensional if and only if it is spanned by exponential monomials.*
Invariant functions and measures

$G$: locally compact group

$K$: compact subgroup with normalized Haar measure $\omega$

$K$-invariant functions in $C(G)$: $f(kxl) = f(x)$ for $x$ in $G$ and $k, l$ in $K$.

These can be identified with the space $C(G//K)$.

$K$-invariant measures in $M_c(K)$: for each $f$ in $C(G)$

$$\int_G f(x) \, d\mu(x) = \int_G \int_K \int_K f(kxl) \, d\omega(k) \, d\omega(l) \, d\mu(x)$$

These can be identified with the functions in the space $M_c(G//K)$, which can be identified with a closed subalgebra of $M_c(K)$. 
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The pair $(G, K)$ is called a Gelfand pair if the algebra $M_c(G//K)$ is commutative.
Invariant functions and measures

\( G \): locally compact group

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\( K \)-invariant functions in \( \mathcal{C}(G) \): \( f(kxl) = f(x) \) for \( x \) in \( G \) and \( k, l \) in \( K \).

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The dual of \( \mathcal{C}(G//K) \) can be identified with \( \mathcal{M}_c(G//K) \).
Invariant functions and measures

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The dual of \( \mathcal{C}(G//K) \) can be identified with \( \mathcal{M}_c(G//K) \).
The projection $f \mapsto f^\#$ on $\mathcal{C}(G)$ is defined as

$$f^\#(x) = \int_K \int_K f(kxl) \, d\omega(k) \, d\omega(l) \quad \text{for} \; x \in G.$$ 

The projection $\mu \mapsto \mu^\#$ on $\mathcal{M}_c(G)$ is defined as

$$\langle \mu^\#, f \rangle = \int_G f^\#(x) \, d\mu(x) \quad \text{for} \; f \in \mathcal{C}(G).$$

Then $f \mapsto f^\#$ is a continuous linear mapping from $\mathcal{C}(G)$ onto $\mathcal{C}(G//K)$ and its adjoint is $\mu \mapsto \mu^\#$:

$$\langle \mu, f^\# \rangle = \langle \mu^\#, f \rangle$$

further $f$ is $K$-invariant if and only if $f = f^\#$ and $\mu$ is $K$-invariant if and only if $\mu = \mu^\#$. 
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László Székelyhidi  
Spectral Synthesis on Affine Groups
Suppose that \((G, K)\) is a Gelfand pair. Then the measures \(\delta_y^#\) commute for all \(y\) in \(G\): for each \(f\) in \(C(G//K)\) we have

\[
\langle \delta_y^# \ast \delta_z^#, f \rangle = \int_K f(ykz) \, d\omega(k) = \int_K f(zky) \, d\omega(k).
\]

Similarly, the operators \(\tau_y\) defined on \(C(G//K)\) by

\[
\tau_y f = \delta_{y^{-1}}^# \ast f = \int f(xz^{-1}) \, d\delta_{y^{-1}}^#(z) = \int_K f(xky) \, d\omega(k)
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form a commuting family for \(y\) in \(G\): these are the \(K\)-translations. \(K\)-translation invariant closed linear subspaces of \(C(G)\) are called \(K\)-varieties.
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form a commuting family for $y$ in $G$: these are the $K$-translations. $K$-translation invariant closed linear subspaces of $C(G)$ are called $K$-varieties. One-dimensional $K$-varieties are spanned by $K$-spherical functions which are the common $K$-invariant eigenfunctions $s$ of all $K$-translations: $\tau_y s = s$ for each $y$ in $G$:

$$\int_K f(xky) \, d\omega(k) = f(x)f(y).$$
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We say that $K$-spectral analysis holds for a $K$-variety if every nonzero $K$-subvariety of it contains a $K$-spherical function. We say that $K$-spectral analysis holds for $G$ if $K$-spectral analysis holds for every $K$-variety.

In a commutative complex algebra $A$ a maximal ideal $M$ is called exponential maximal ideal, if $A/M$ is isomorphic to the complex field.

$K$-spectral analysis

$K$-spectral analysis holds for the $K$-variety $V$ if and only if for every closed maximal ideal $M$ of the residue algebra $\mathcal{M}_c(G//K)/\text{Ann } V$ is exponential. $K$-spectral analysis holds for $G$ if and only if every closed maximal ideal of $\mathcal{M}_c(G//K)$ is exponential.
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In a commutative complex algebra $A$ a maximal ideal $M$ is called \textit{exponential maximal ideal}, if $A/M$ is isomorphic to the complex filed.
Modified differences and $K$-monomials

For each $K$-spherical function $s$ we define the modified $K$-difference

$$\Delta_{s; y} = \delta_y^* - s(y)\delta_e,$$

and their products $\Delta_{s; y_1, y_2, \ldots, y_{k+1}} = \prod_{j=1}^{k+1} \Delta_{s; y_j}$. Given the $K$-spherical function $s$ the closed ideal generated by all modified differences $\Delta_{s; y}$ with $y$ in $G$ is an exponential maximal ideal, denoted by $M_s$. The $K$-invariant $f$ is called an $s$-monomial if $\dim \tau_K(f) < \infty$ is and there is a natural number $k$ such that

$$M_s^{k+1} \subseteq \text{Ann} \tau_K(f)$$

where $\tau_K(f)$ denotes the $K$-variety generated by $f$. This is equivalent to the functional equation

$$\Delta_{s; y_1, y_2, \ldots, y_{k+1}} \ast f(x) = 0$$

for each $x, y_1, y_2, \ldots, y_{k+1}$ in $G$. If $f$ is nonzero, then $s$ is uniquely determined, and the smallest $k$ with this property is called the degree of the $s$-monomial $f$. 

László Székelyhidi
Spectral Synthesis on Affine Groups
For instance, \( s \)-monomials of degree 2 are of the form \( cs + f \), where \( f \) is a \( K \)-invariant continuous solutions of the \( K \)-sine equation:

\[
\int_K f(xky) \, d\omega(k) = f(x)s(y) + f(y)s(x).
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$$\int_K f(xky) \, d\omega(k) = f(x)s(y) + f(y)s(x).$$

We say that the $K$-variety is $K$-synthesizable if all $K$-monomials span a dense subspace in the variety. We say that $K$-spectral synthesis holds for a $K$-variety, if every nonzero subvariety of it is $K$-synthesizable.

$K$-spectral synthesis implies $K$-spectral analysis.
K-spectral synthesis

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**K-synthesizability**

The $K$-variety $V$ is synthesizable if and only if its annihilator is the intersection of those cofinite closed ideals of $M_c(G//K)$ which contain it.
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If $K$ is a normal subgroup and $G/K$ is commutative, then all these concepts coincide with the corresponding spectral analysis and synthesis concepts on the locally compact Abelian group $G/K$. Obviously, this is the case if $G$ itself is commutative.
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Semidirect products

Let $N$ be a locally compact topological group and $K$ is a compact group of automorphisms of $N$. We consider the semidirect product of $K$ and $N$: $K \ltimes N$: it is $K \ltimes N$ equipped with the operation

$$(k, n) \cdot (l, m) = (k \circ l, (k \cdot m)n),$$

where $\circ$ is the composition of the automorphisms $k, l$, $\cdot$ is the effect of the automorphisms on the elements of $N$, and juxtaposition is the group operation in $N$. It turns out that this operation defines a group structure on $K \ltimes N$, where the identity is $(id, e)$, with the identity automorphism $id$ of $N$ and the identity element $e$ of $n$, and the inverse of $(k, n)$ is $(k^{-1}, k^{-1} \cdot u^{-1})$. With the product topology $G = K \ltimes N$ is a locally compact topological group, the semidirect product of $K$ and $N$. The group $N$ is topologically isomorphic to the closed normal subgroup $\{(id, n) : n \in N\}$, and the group $K$ is topologically isomorphic to the compact subgroup $\{(k, e) : k \in K\}$. We shall identify these isomorphic groups:

$$K = \{(k, e) : k \in K\}, \quad N = \{(id, n) : n \in N\}.$$
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$$K = \{(k, e) : k \in K\}, \quad N = \{(id, n) : n \in N\}.$$
Example: Affine groups

Let $X$ be a finite dimensional vector space and $K$ a compact subgroup of $GL(X)$, the general linear group of $X$ with the normalized Haar measure $\omega$. Then the set $K \times X$ acts on $X$: for $S$ in $K$ and $u$ in $V$ let $(S, u)x$ defined by the affine mapping

$$(S, u)x = Sx + u$$

for each $x$ in $V$. The composition of affine mappings defines the operation on $K \times X$ as

$$(S, u) \cdot (T, v) = (S \circ T, Sv + u)$$

and with the identity $(id, 0)$ and inverse $(S, u)^{-1} = (S^{-1}, -S^{-1}u)$ we obtain the group

$$\text{Aff } K = K \ltimes X,$$

the semidirect product of $K$ and $X$. Here – as we have seen – $K$ is topologically isomorphic to the compact subgroup $\{(S, 0) : S \in K\}$ and $X$ is topologically isomorphic to the closed normal subgroup $\{(id, u) : u \in X\}$. 

Example: Semidirect products

\( K \)-invariant functions are exactly those functions \((S, u) \mapsto f(S, u)\) which depend only on \(u\) and are invariant with respect to \(K\):

\[
f(S, u) = f(id, u) = f(id, Su)
\]

for each \(S\) in \(K\) and \(u\) in \(X\). Hence \(C(\text{Aff } K//K)\) can be identified with a closed subspace of \(C(X)\), the space of \(K\)-radial functions. Similarly, the space of \(K\)-invariant measures \(M_c(\text{Aff } K//K)\) on \(\text{Aff } K\) can be identified with a closed subspace of \(M_c(X)\), the space of \(K\)-radial measures,

Then \(\text{Aff } K\) is a locally compact group, \(K\) is topologically isomorphic to the compact subgroup \(\{(L, 0) : L \in K\}\), and \(\mathbb{R}^n\) is topologically isomorphic to the normal subgroup \(\{\text{id}, u) : u \in \mathbb{R}^n\}\).
$K$-invariant functions are exactly those functions $(S, u) \mapsto f(S, u)$ which depend only on $u$ and are invariant with respect to $K$:

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Then $\text{Aff } K$ is a locally compact group, $K$ is topologically isomorphic to the compact subgroup $\{(L, 0) : L \in K\}$, and $\mathbb{R}^n$ is topologically isomorphic to the normal subgroup $\{(id, u) : u \in \mathbb{R}^n\}$. 
The Poincaré group

We consider the real vector space $\mathbb{R}^{1,3} = \mathbb{R} \oplus \mathbb{R}^3$ equipped with the *indefinite inner product*

$$\langle v, w \rangle = v_0 w_0 - \sum_{j=1}^{3} v_j w_j,$$

where $v = (v_0, v_1, v_2, v_3)$ and $w = (w_0, w_1, w_2, w_3)$. The *isometry group* $O(1, 3)$ of this indefinite inner product space is called the *Lorentz group*. The affine group of the Lorentz group

$$\text{Aff } O(1, 3) = O(1, 3) \ltimes \mathbb{R}^{1,3}$$

is the *Poincaré group*. 
The Poincaré group

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\]

is the Poincaré group.
The group of Euclidean motions

We consider the vector space $\mathbb{R}^n$ and the orthogonal group $O(n)$, the group of rotations. Together with translations they generate the group of Euclidean motions: rigid motions leaving the origin fixed. This is the affine group of $O(n)$:

$$\text{Aff } O(n) = O(n) \times \mathbb{R}^n$$

which acts on $\mathbb{R}^n$ by

$$(O, u)x = Ox + u$$

for $O$ in $O(n)$ and $x, u$ in $\mathbb{R}^n$.

Clearly, for $n = 1$ we have $O(1) = \{+1, -1\}$. $O(1)$-spherical functions are the functions of the form $x \mapsto \cosh \lambda x$ with arbitrary complex $\lambda$. 
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The group of proper motions

We consider the vector space $\mathbb{R}^n$ and the *special orthogonal group* $SO(n)$, the group of proper rotations: orthogonal operators with determinant $+1$. Together with translations they generate the group of *proper Euclidean motions*: rigid motions which preserve orientation: no reflection is included. This is the affine group of $SO(n)$:

$$\text{Aff } SO(n) = SO(n) \times \mathbb{R}^n$$

which acts on $\mathbb{R}^n$ by

$$(S, u)x = Sx + u$$

for $S$ in $SO(n)$ and $x, u$ in $\mathbb{R}^n$.

Clearly, for $n = 1$ we have $SO(1) = \{id\}$, hence $\text{Aff } SO(1) = \mathbb{R}$ — in one dimension the proper Euclidean motions are exactly the translations.
Example: Proper Euclidean motions

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Example: Proper Euclidean motions

The $SO(n)$-invariant functions can be identified with those continuous functions $f : \mathbb{R}^n \to \mathbb{C}$ with $f(Sx) = f(x)$ for each $S$ in $SO(n)$ and $x$ in $\mathbb{R}^n$. These are called radial functions as $f(x)$ depends only on $\|x\|$:

$$f(x) = \varphi(\|x\|)$$

for some continuous $\varphi : \mathbb{R} \to \mathbb{C}$. Similarly, $\mathcal{M}_c(\text{Aff } SO(n))$ is identified with those measures in $\mathcal{M}_c(\mathbb{R}^n)$ with

$$\int_{\mathbb{R}^n} f(Sx) \, d\mu(x) = \int_{\mathbb{R}^n} f(x) \, d\mu(x)$$

for each $f$ in $\mathcal{C}(\mathbb{R}^n)$ and $S$ in $SO(n)$: radial measures.
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$$\int_{\mathbb{R}^n} f(Sx) \, d\mu(x) = \int_{\mathbb{R}^n} f(x) \, d\mu(x)$$

for each $f$ in $C(\mathbb{R}^n)$ and $S$ in $SO(n)$: radial measures. Convolution in $\mathcal{M}_c(\text{Aff } SO(n) // SO(n))$ coincides with the ordinary convolution in $\mathbb{R}^n$, hence $(\text{Aff } SO(n), SO(n))$ is a Gelfand pair.

Radial functions: $C_r(\mathbb{R}^n) \approx C(\text{Aff } (SO(n)) // SO(n))$

Radial measures: $\mathcal{M}_r(\mathbb{R}^n) \approx \mathcal{M}_c(\text{Aff } (SO(n)) // SO(n))$
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Example: Proper Euclidean motions

**SO(n)-translation:** for \( f \) in \( C_r(\mathbb{R}^n) \) and \( y \) in \( \mathbb{R}^n \)

\[
\tau_y(f)(x) = \int_{SO(n)} f(x + ky) \, d\omega(k)
\]

**SO(n)-variety:** \( V \subseteq C_r(\mathbb{R}^n) \) linear subspace, closed with respect to uniform convergence on compact sets, and for each \( f \) in \( V \) and \( y \) in \( \mathbb{R}^n \) we have \( x \mapsto \int_{SO(n)} f(x + ky) \, d\omega(k) \) is in \( V \)

**SO(n)-spherical function:** \( s \neq 0 \) in \( C_r(\mathbb{R}^n) \) and

\[
\int_{SO(n)} s(x + ky) \, d\omega(k) = s(x) s(y) \quad \text{for each} \quad y \in \mathbb{R}^n
\]
Example: Proper Euclidean motions

Eigenfunctions of the Laplacian

The $SO(n)$-spherical functions are exactly the normalized radial eigenfunctions of the Laplacian in $\mathbb{R}^n$.

Let

$$\phi(\|x\|) = s(x) \quad \text{for} \quad x \in \mathbb{R}^n,$$

then, using the radial form of the Laplacian in $\mathbb{R}^n$ we have the Bessel differential equation

$$\frac{d^2}{dr^2} \phi(r) + \frac{n-1}{r} \frac{d}{dr} \phi(r) = \lambda \phi(r),$$

with $\phi$ is regular at 0 and $\phi(0) = 1$. Let $J_\lambda$ denote the function

$$J_\lambda(r) = \Gamma\left(\frac{n}{2}\right) \sum_{k=0}^{\infty} \frac{\lambda^k}{k! \Gamma(k + \frac{n}{2})} \left(\frac{r}{2}\right)^{2k}.$$

Then $s$ is an $SO(n)$-spherical function if and only if

$$s(x) = s_\lambda(x) = J_\lambda(\|x\|)$$

holds for each $x$ in $\mathbb{R}^n$ with some complex number $\lambda$. 
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SO(n)-monomials

Derivatives with respect to the parameter

Given the SO(n)-spherical function $s_\lambda$ with some complex $\lambda$ the $s_\lambda$-monomials of degree at most $k$ are exactly the linear combinations of the derivatives $\frac{d^j}{d\lambda^j} s_\lambda$ for $j = 0, 1, \ldots, k$.

SO(n)-spectral analysis and synthesis

Every nonzero variety contains an SO(n)-spherical function, moreover, all functions of the form $\frac{d^j}{d\lambda^j} s_\lambda$ span a dense subspace in every variety.
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As $SO(1) = \{id\}$, hence $\text{Aff } SO(1) = \mathbb{R}$, $SO(1)$-varieties are exactly the closed translation invariant subspaces of $C(\mathbb{R})$. $SO(1)$-spherical functions are exactly the exponentials: $s_\lambda(x) = e^{\lambda x}$, and $SO(1)$-monomials are the linear combinations of the functions

$$\frac{d^j}{d\lambda^j} s_\lambda(x) = x^j e^{\lambda x}.$$

Our spectral synthesis theorem is a proper generalization of L. Schwartz’s theorem to $\mathbb{R}^n$. 
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