On the Images of Sobolev space under Schrodinger semigroup associated to the Dunkl operator

Sivaramakrishnan C

IIT Hyderabad

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University of pecs.
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Overview

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Classical settings

- For ∂_i $(i = i, 2, \cdots, n)$ and Δ we have semigroups on $L^2(\mathbb{R}^n, du)$. Such as Heat kernel semigroup, Hermite semigroup, Special Hermite semigroup, Schrodinger semigroup and so on...
- The Fourier transform $\mathcal{F} : L^2(\mathbb{R}, du) \to L^2(\mathbb{R}, du)$ is unitary.
- For $x \in \mathbb{R}^n$ the translation operator $\tau_x : L^2(\mathbb{R}, du) \to L^2(\mathbb{R}, du)$. Where $\tau_x f(y) = f(x - y)$.
- Convolution ...

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- an operator which is having properties similar to the Differential operator.
- a transformation which is having properties similar to the Fourier transform.

Basics

- C. F. Dunkl, Reflection groups and orthogonal polynomials on the sphere, Math. Z. **197**, 33-60(1988).
- C. F. Dunkl, Differential-difference operators associated to reflection groups, Trans. Amer. Math. Soc. **311** (1989)no. 1.

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For $\mu > 0$, the Dunkl operator associated to the reflection group $\mathbb{Z}_2$ is denoted by $\mathcal{D}_\mu$ and it is given by

Dunkl Operator

$$(\mathcal{D}_\mu f)(x) = \frac{df}{dx}(x) + \frac{\mu}{x}(f(x) - f(-x)), x \in \mathbb{R}.$$

Theorem

Now consider the equation, for $x, y \in \mathbb{R}$,

$$\mathcal{D}_\mu f(x,y) = yf(x,y).$$

The above equation has a unique real analytic solution $E_\mu : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ *and it can be extended as an analytic function* $E_\mu : \mathbb{C} \times \mathbb{C} \to \mathbb{C}$.

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Definition

The function $E_\mu(x, y)$ is called the Dunkl kernel.

Basics

Dunkl Kernel

$$E_\mu(x, y) = \sum_{k=0}^{\infty} \frac{(xy)^k}{\gamma_\mu(k)}.$$

Where for $k \in \mathbb{N}$,

Generalized factorial function

$$\gamma_\mu(2k) = \frac{2^{2k} k! \Gamma(k + \mu + \frac{1}{2})}{\Gamma(\mu + \frac{1}{2})} \text{ and } \gamma_\mu(2k+1) = \frac{2^{2k+1} k! \Gamma(k + \mu + \frac{3}{2})}{\Gamma(\mu + \frac{1}{2})}.$$

Basics

Dunkl transform

For $f \in L^1(\mathbb{R}, |u|^{2\mu} du)$, the Dunkl transform of f is defined by,

$$\hat{f}(y) = c_\mu^{-1} \int_{\mathbb{R}} f(x) E_\mu(-ix, y) |u|^{2\mu} dx, y \in \mathbb{R}^n.$$

Where c_μ is the constant chosen so that $c_\mu = \int_{\mathbb{R}} e^{-\frac{|x|^2}{2}} |u|^{2\mu} dx$.

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Theorem

The Dunkl transform : $L^2(\mathbb{R}, |u|^{2\mu} du) \to L^2(\mathbb{R}, |u|^{2\mu} du)$ *is unitary.*

Basics

Dunkl Translation

The generalized translation (or Dunkl translation) of a function $f \in L^2(\mathbb{R}, |u|^{2\mu} du)$ is defined by

$$\tau_y^\mu f(x) = c_\mu^{-1} \int_{\mathbb{R}} \hat{f}(\xi) E_\mu(ix, \xi) E_\mu(-iy, \xi) |\xi|^{2\mu} d\xi, x, y \in \mathbb{R}.$$

Genrealized convolution

Generalized convolution of $f, g \in L^2(\mathbb{R}, |u|^{2\mu} du)$ is given by

$$f *_\mu g(x) = \int_{\mathbb{R}} f(y) \tau_x^\mu \check{g}(y) |y|^{2\mu} dy,$$

where $\check{g}(u) = g(-u)$. Equivalently it can be written as

$$f *_\mu g(x) = \int_{\mathbb{R}} \hat{f}(\xi) \hat{g}(\xi) E_\mu(ix, \xi) |\xi|^{2\mu} d\xi.$$

Heat semi group

Dunkl Laplacian on $\mathbb{R}^n$

$$\Delta_\mu := \mathcal{D}_\mu^2.$$

Theorem

Dunkl Laplacian generates a strongly continuous, positive prserving semigroup on $L^2(\mathbb{R}, |u|^{2\mu} du)$. Where,

$$e^{t\Delta_\mu} f(x) := \begin{cases} \int_{\mathbb{R}} f(y)\Gamma_\mu(t,x,u)|u|^{2\mu} du & \text{if } t > 0 \\ f & \text{if } t = 0. \end{cases}$$

and $\Gamma_\mu(t,x,y) = \frac{c_\mu^{-1} 2^{-(\mu+\frac{n}{2})}}{t^{\mu+\frac{n}{2}}} e^{-\frac{|x|^2+|y|^2}{4t}} E_\mu(\frac{x}{\sqrt{2t}}, \frac{y}{\sqrt{2t}}),\ x, y \in \mathbb{R}.$

Note

$\Gamma_\mu(t,x,y) = \tau_y^\mu F_\mu(t,x)$, where $F_\mu(t,x) = \frac{c_\mu^{-1} 2^{-(\mu+\frac{1}{2})}}{t^{\nu_\mu+\frac{n}{2}}} e^{-\frac{x^2}{4t}}$.

Heat Semigroup

- $e^{t\Delta_\mu} : L^2(\mathbb{R}, |u|^{2\mu} du) \to L^2(\mathbb{R}, |u|^{2\mu} du)$ is injective bounded operator.
- $e^{t\Delta_\mu} f = f * F_\mu(t, .)$, for $f \in L^2(\mathbb{R}, |u|^{2\mu} du)$. So $e^{t\Delta} f$ can be extended as an entire function on $\mathbb{C}$.
- $e^{t\Delta_\mu} : L^2(\mathbb{R}, |u|^{2\mu} du) \to \mathcal{O}(\mathbb{C})$, where $\mathcal{O}(\mathbb{C})$ is the space of all analytic functions on $\mathbb{C}$.
- Consider $e^{t\Delta_\mu}(L^2) = \left\{ e^{t\Delta_\mu} f : f \in L^2(\mathbb{R}, |u|^{2\mu} du) \right\}$.

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- $e^{t\Delta_\mu} : L^2(\mathbb{R}, |u|^{2\mu} du) \to \mathcal{O}(\mathbb{C})$, where $\mathcal{O}(\mathbb{C})$ is the space of all analytic functions on $\mathbb{C}$.
- Consider $e^{t\Delta_\mu}(L^2) = \left\{ e^{t\Delta_\mu} f : f \in L^2(\mathbb{R}, |u|^{2\mu} du) \right\}$.
- For $f, g \in L^2(\mathbb{R}, |u|^{2\mu} du)$, define $\langle e^{t\Delta_\mu} f, e^{t\Delta_\mu} g \rangle_{e^{t\Delta_\mu}(L^2)} := \langle f, g \rangle_{L^2(\mathbb{R}, |u|^{2\mu} du)}$.
- With respect to above inner product $e^{t\Delta_\mu}(L^2)$ becomes a Hilbert space and $e^{t\Delta_\mu} : L^2(\mathbb{R}, |u|^{2\mu} du) \to e^{t\Delta_\mu}(L^2)$ is unitary.

Heat semigroup

- Is there exists a a positive continuous function $\rho(z)$ on $\mathbb{C}$ such that

$$\langle e^{t\Delta_\mu} f, e^{t\Delta_\mu} g\rangle_{e^{t\Delta_\mu}(L^2)} = \int_{\mathbb{C}} e^{t\Delta_\mu} f(z)\overline{e^{t\Delta_\mu} g(z)}\rho(z)dz$$

  for all $f, g \in L^2(\mathbb{C}, |u|^{2\mu} du)$.

- Weighted Bergman space:

$$\mathcal{HL}^2_\rho := \mathcal{HL}^2(\mathbb{C}, \rho(z)dz) = \left\{ F \in \mathcal{O}(\mathbb{C}) : \int_{\mathbb{C}} |F(z)|^2 \rho(z)dx < \infty \right\}.$$

  The space $\mathcal{HL}^2(\mathbb{C}, \rho(z)dz)$ becomes a Hilber space with respect to the following inner product: $\langle F, G\rangle_{\mathcal{HL}^2_\rho} := \int_{\mathbb{C}} F(z)\overline{G(z)}\rho(z)dz$.

-
- For $\mu = 0$,

$$e^{t\Delta}(L^2) = \mathcal{HL}^2(\mathbb{C}, (4\pi t)^{-\frac{1}{2}} e^{-\frac{y^2}{4t}} dz).$$

Heat semigroup

Theorem

The operator $e^{t\Delta_\mu} : L^2(\mathbb{R}, |u|^{2\mu} du) \to \mathcal{C}_{\mu,t}$ *is unitary. Where* $\mathcal{C}_{\mu,t}$ *is the Hilbert space of analytic functions on* $\mathbb{C}$ *with reproducing kernel*

$$\mathbb{K}_{\mu,t}(z, w) := c_\mu e^{-\left(\frac{z^2+\overline{w}^2}{8t}\right)} E_\mu\left(\frac{z}{2t^{\frac{1}{2}}}, \frac{\overline{w}}{2t^{\frac{1}{2}}}\right), \;\; z, w \in \mathbb{C}.$$

S. B. Sontz, The μ-deformed Segal-Bargmann transform is a Hall type transform, Infin. Dimens. Anal. Quantum Probab. Relat. Top. **12** (2009)no. 2.

S. B. Sontz, On Segal-Bargmann analysis for finite Coxeter groups and its heat kernel, Math. Z. **269** (2011)no. 1-2.

Given any $f \in \mathcal{O}(\mathbb{C})$ it can be written as $f = f_e + f_o$.

$$\mathcal{C}_{\mu,t} := \{f \in \mathcal{O}(\mathbb{C}) : f_e \in L^2(\mathbb{C}, \nu_{e,\mu,t}(z)dz) \text{ and } f_o \in L^2(\mathbb{C}, \nu_{o,\mu,t}(z)dz)\}.$$

Where, For $z \in \mathbb{C}, t > 0$ and $\mu > 0$, define the weight functions

$$\nu_{e,\mu,t}(z) := \pi^{-1} 2^{\frac{1}{2}+\mu} (2t)^{\mu-\frac{1}{2}} e^{\frac{z^2+\bar{z}^2}{8t}} K_{\mu-\frac{1}{2}}(|\frac{z}{(4t)^{\frac{1}{2}}}|^2) |\frac{z}{(4t)^{\frac{1}{2}}}|^{2\mu+1} \tag{1}$$

$$\nu_{o,\mu,t}(z) := \pi^{-1} 2^{\frac{1}{2}+\mu} (2t)^{\mu-\frac{1}{2}} e^{\frac{z^2+\bar{z}^2}{8t}} K_{\mu+\frac{1}{2}}(|\frac{z}{(4t)^{\frac{1}{2}}}|^2) |\frac{z}{(4t)^{\frac{1}{2}}}|^{2\mu+1}. \tag{2}$$

The space $\mathcal{C}_{\mu,t}$ is the Hilbert space with respect to the following inner product: For $f, g \in \mathcal{C}_{\mu,t}$,

$$\langle f, g \rangle_{\mathcal{C}_{\mu,t}} := \langle f_o, g_o \rangle_{L^2(\mathbb{C}, \nu_{o,\mu,t}(z)dz)} + \langle f_e, g_e \rangle_{L^2(\mathbb{C}, \nu_{e,\mu t}(z)dz)}.$$

Schrodinger semi group

Image

- Δ_μ is self-adjoint.
- The operator $i\Delta_\mu$ is skew-adjoint.
- By Stones theorem, $i\Delta_\mu$ generates a strongly continuous unitary semigroup $\left(e^{it\Delta_\mu}\right)_{t\geq 0}$ on $L^2(\mathbb{R}^n, |u|^{2\mu}du)$.
- Where,

$$e^{it\Delta_\mu} f := \begin{cases} \int_{\mathbb{R}^n} \Gamma_\mu(it, ., y) f(y) |u|^{2\mu} dy & \text{if } t > 0 \\ f & \text{if } t = 0. \end{cases}$$

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Moreover, $e^{it\Delta_\mu}f$ solves the Schrodinger equation associated to the Dunkl Laplacian Δ_μ on $\mathbb{R}^n$

$$i\partial_t u \;=\; \Delta_\mu u \;;\; u(x,0) = f(x),$$

where $f \in L^2(\mathbb{R}, |u|^{2\mu}du)$.

Image of L^2_μ Schrodinger Semi group

- The Schrodinger semigroup is unitary on $L^2(\mathbb{R}, |u|^{2\mu} du)$.
- $e^{it} f$ cannot be extended as an entire function on $\mathbb{C}$ for all $f \in L^2(\mathbb{R}, |u|^{2\mu} du)$.

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- So, for $f \in L^2_\mu := L^2(\mathbb{R}, e^{u^2} |u|^{2\mu} du)$, it is easy to see that $e^{it\Delta_\mu} f$ can be extended as an analytic function on $\mathbb{C}$.

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- So, for $f \in L^2_\mu := L^2(\mathbb{R}, e^{u^2} |u|^{2\mu} du)$, it is easy to see that $e^{it\Delta_\mu} f$ can be extended as an analytic function on $\mathbb{C}$.
- Hence $e^{it\Delta_\mu}(L^2_\mu)$ is a subspace of space of all analytic functions on $\mathbb{C}$.
- It is clear that $e^{it\Delta_\mu} : L^2(\mathbb{R}, e^{u^2} |u|^{2\mu} du) \longrightarrow e^{it\Delta_\mu}(L^2_\mu)$ is linear and bijective.
- For $f, g \in L^2(\mathbb{R}, e^{u^2} |u|^{2\mu} du)$ we define,

$$\langle e^{it\Delta_\mu} f, e^{it\Delta_\mu} g \rangle_{e^{it\Delta_\mu}(L^2_\mu)} := \langle f, g \rangle_{L^2_\mu},$$

- With respect to the above inner product $e^{it\Delta_\mu}(L^2_\mu)$ becomes a Hilbert space. Our aim is to identify this space as a weighted Bergman space.

Image of L^2_μ Schrodinger Semi group

For $\mu = 0$ case

$e^{it\Delta} : L^2(\mathbb{R}, e^{u^2} du) \longrightarrow \mathcal{HL}^2(\mathbb{C}, w_t(x+iy)dxdy)$ is unitary, where $w_t(x+iy) = \frac{1}{(2\sqrt{\pi t})} e^{\frac{xy}{t} - \frac{y^2}{4t^2}}$.

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Image of L^2_μ Schrodinger Semi group

- For $f \in L^2_\mu$ set $g(u) = f(u)e^{\frac{u^2}{2}} e^{\frac{i}{4t}u^2}$, we have

$$e^{it\Delta_\mu} f(z) \quad = \frac{1}{(2it)^{\mu+\frac{1}{2}}} e^{\frac{i}{4t}z^2} \quad \left(\hat{g} *_\mu F_\mu(\frac{1}{2}, .)\right) (\frac{z}{2t}), \forall z \in \mathbb{C}.$$

- Define a linear map $\mathcal{G} : \mathcal{O}(\mathbb{C}) \longrightarrow \mathcal{O}(\mathbb{C})$ by $\mathcal{G}F(z) = (it)^{\nu_\mu + \frac{n}{2}} e^{-itz^2} F(2tz)$.
- Consider the space,

$$\mathcal{H}_{\mu,t} := \left\{ F \in \mathcal{O}(\mathbb{C}) : \mathcal{G}(F) \in \mathcal{C}_{\mu,\frac{1}{2}} \right\}.$$

- The space $\mathcal{H}_{\mu,t}$ is Hilbert space with respect to the following inner product : For $F, G \in \mathcal{H}_{\mu,t}$,

$$\langle F, G \rangle_{\mathcal{H}_{\mu,t}} \quad := \quad \langle \mathcal{G}F, \mathcal{G}G \rangle_{\mathcal{C}_{\mu,\frac{1}{2}}}.$$

Image of L^2_μ Schrodinger Semi group

Theorem

The operator $e^{it\Delta_\mu} : L^2(\mathbb{R}, |u|^{2\mu} e^{u^2} du) \longrightarrow \mathcal{H}_{\mu,t}$ *is unitary.*

Note

The space $\mathcal{H}_{\mu,t}$ is a Reproduing Kernel Hilbert space with Reproducing kernel is given by,

$$\overline{\mathbb{K}}_{\mu,t}(w,z) = |(2ti)^{-(\mu+\frac{1}{2})}|^2 e^{\frac{i}{4t}(-\bar{z}^2+w^2)} \mathbb{K}_{\mu,\frac{1}{2}}\left(\frac{w}{2\sqrt{2}t}, \frac{\bar{z}}{2\sqrt{2}t}\right).$$

Sobolev space

For $m \in \mathbb{N}$,

Dunkl Sobolev space

- $$W_\mu^{m,2}(\mathbb{R}) := \left\{ f \in L_\mu^2 : \mathcal{D}_\mu^k f \in L_\mu^2, \text{ for } k \in \mathbb{N} \text{ with } k \leq m \right\}. \tag{3}$$
- For $f, g \in W_\mu^{m,2}(\mathbb{R})$, define
$$\langle f, g \rangle_{W_\mu^{m,2}} := \sum_{k=0}^{m} \langle \mathcal{D}_\mu^k f, \mathcal{D}_\mu^k g \rangle_{L_\mu^2}.$$

- The space $W_\mu^{m,2}(\mathbb{R})$ becomes a Hilbert space with respect to the above inner product.

Image of Dunkl Sobolev space under Schrodinger Semigroup

- Define,

$$e^{it\Delta_\mu}(W_\mu^{m,2}) := \left\{ e^{it\Delta_\mu} f \in \mathcal{O}(\mathbb{C}) : f \in W_\mu^{m,2}(\mathbb{R}^n) \right\}.$$

- $e^{it\Delta_\mu} : W_\mu^{m,2}(\mathbb{R}) \longrightarrow e^{it\Delta_\mu}(W_\mu^{2,m})$ is a bijective linear map.
- For $f, g \in W_\mu^{m,2}(\mathbb{R}^n)$,

$$\left\langle e^{it\Delta_\mu} f, e^{it\Delta_\mu} g \right\rangle_{e^{it\Delta_\mu}(W_\mu^{2,m})} := \langle f, g \rangle_{W_\mu^{m,2}}.$$

- With respect to the above inner product, the space $e^{it\Delta_\mu}(W_\mu^{2,m})$ becomes a Hilbert space.
- The operator $e^{it\Delta_\mu} : W_\mu^{m,2}(\mathbb{R}) \longrightarrow e^{it\Delta_\mu}(W_\mu^{m,2})$ becomes unitary.

Image of Dunkl Sobolev space under Schrodinger Semigroup

- We cannot get a single weight function on $\mathbb{C}$ such that $e^{it\Delta_\mu}(W_\mu^{m,2})$ becomes a Hilbert space such that $e^{it\Delta_\mu} : W_\mu^{m,2} \to e^{it\Delta_\mu}(W_\mu^{m,2})$ becomes unitary.

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- So we are looking inner product on $e^{it\Delta_\mu}(W_\mu^{m,2})$ such that $e^{it\Delta_\mu}(W_\mu^{m,2})$ becomes a Hilbert space such that $e^{it\Delta_\mu} : W_\mu^{m,2} \to e^{it\Delta_\mu}(W_\mu^{m,2})$ becomes bounded invertible map.

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- So we are looking inner product on $e^{it\Delta_\mu}(W_\mu^{m,2})$ such that $e^{it\Delta_\mu}(W_\mu^{m,2})$ becomes a Hilbert space such that $e^{it\Delta_\mu} : W_\mu^{m,2} \to e^{it\Delta_\mu}(W_\mu^{m,2})$ becomes bounded invertible map.
- That is Image of $W_\mu^{m,2}$ under Schrodinger semigroup is norms equivalent some weighted Bergman space.

Image of sobolev space in $L^2(\mathbb{R}, e^{u^2}du)$ under Scrodinger semigroup I

Let $u_t^m(z) = \sum_{k=0}^{m} \left(\frac{1}{t}\right)^{2k} |z^k|^2 w_t(z)$ and consider the weighted Bergman space $\mathcal{HL}^2(\mathbb{C}, u_t^m(z)dz)$.

Theorem

The operator $e^{it\Delta} : W^{m,2}(\mathbb{R}) \longrightarrow \mathcal{HL}^2(\mathbb{C}, u_t^m(z)dz)$ is bounded invertible.

Proof:

- Hermite polynomial: For k $\mathbb{N}$, $H_k(x) = k! \sum_{k=0}^{[\frac{k}{2}]} \frac{(-1)^j (2x)^{k-2j}}{j!(k-2j)!}$.
  Consider $\psi_k(u) = \left(\frac{1}{\sqrt{\pi}2^k k!}\right)^{\frac{1}{2}} H_k(u) e^{-u^2}, u \in \mathbb{R}$.
- The set $\{\frac{\psi_k}{\|\psi_k\|_{W^{m,2}(\mathbb{R})}} : k \in \mathbb{N}\}$ forms an orthonormal basis for $W^{m,2}(\mathbb{R})$.

Image of sobolev space in $L^2(\mathbb{R}, e^{u^2}du)$ under Scrodinger semigroup II

- Consider the multiplication operator $\mathcal{S}f(u) = f(u)e^{-\frac{i}{4t}u^2}$ for $f \in w^{m,2}(\mathbb{R})$.
- $\mathcal{S} : W^{m,2}(\mathbb{R}) \longrightarrow W^{m,2}(\mathbb{R})$ is bounded invertible map.
- $e^{it\Delta_\mu}\mathcal{S}\Psi_k^\mu(z) = i^{-\frac{1}{2}}\left(\frac{1}{i}\right)^{|\alpha|}\Upsilon_\alpha^t$ for all $k \in \mathbb{N}$ and $z \in \mathbb{C}$. Where $\Upsilon_k^t(z) = (4i\pi t)^{-\frac{1}{2}}(\sqrt{\pi})^{\frac{1}{2}}\left(\frac{1}{2^{|k|}k!}\right)^{\frac{1}{2}}\left(\frac{1}{2ti}\right)^{|k|} z^k e^{(\frac{i}{4t}-\frac{1}{16t^2})z^2}$.
- The set $\left\{\frac{\Upsilon_k}{\|\Upsilon_k\|_{\mathcal{HL}^2_{u_t^m}}} : k \in \mathbb{N}\right\}$ form an orthonormal basis for $\mathcal{HL}^2(\mathbb{C}, u_t^m(x+iy)dxdy)$.
- $\|\Upsilon_k\|_{\mathcal{HL}^2_{u_t^m}} = \|\psi_k\|_{W^{m,2}(\mathbb{R})}$. for all $k \in \mathbb{N}$.
- $e^{it\Delta}\mathcal{S} : W^{m,2}(\mathbb{R}) \to \mathcal{HL}^2(\mathbb{C}, u_t^m(x+iy)dxdy)$ is unitary.
- $\mathcal{HL}^2(\mathbb{C}, u_t^m(x+iy)dxdy)$ is equivalent to the space $e^{it\Delta}(W^{m,2})$.

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Thanks for your attention...
Köszönöm...