Rational Fourier – Chebyshev series of some elementary functions

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1. Introduction

Chebyshev polynomials:

$$T_n(x) = \cos n \arccos x, \quad U_n(x) = \frac{\sin(n+1)\arccos x}{\sqrt{1-x^2}}$$

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$\{a_k\}_{k=1}^{2n}$:
1) $a_k \in \mathbb{R}$, $|a_k| < 1$; 2) paired by complex conjugation.

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Angles $(k = 1, 2, \ldots, 2n)$

$$\cos \varphi_k = \frac{\sqrt{(1 + a_k)(1 + x)}}{\sqrt{2(1 + a_k x)}}, \quad \sin \varphi_k = \frac{\sqrt{(1 - a_k)(1 - x)}}{\sqrt{2(1 + a_k x)}}. \quad (1)$$

Sum

$$\Phi_n(x) = \sum_{k=1}^{2n} \varphi_k(x), \quad x \in [-1, 1]. \quad (2)$$

Chebyshev – Markov rational cosine and sine-fractions:

$$\cos \Phi_n(x), \qquad \sin \Phi_n(x).$$

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Special case:

$$a_k = ia,\ k = 1, 2, \ldots, n; \quad a_k = -ia,\ k = n+1, n+2, \ldots, 2n,$$
$$a \in [0, +\infty).$$

Chebyshev – Markov cosine-fraction on the segment $[-1, 1]$ with two complex conjugate poles:

$$M_n(x) = \cos n \arccos \left(x \frac{\sqrt{1+a^2}}{\sqrt{1+a^2x^2}} \right), \quad n = 0, 1, \ldots . \qquad (3)$$

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2. Chebyshev – Markov rational fractions. Lemma 1

The following formula holds:

$$M_n(x) = \frac{1}{2}\left(\left(\frac{\sqrt{1+a^2}x + i\sqrt{1-x^2}}{\sqrt{1+a^2x^2}}\right)^n + \right.$$
$$\left. + \left(\frac{\sqrt{1+a^2}x - i\sqrt{1-x^2}}{\sqrt{1+a^2x^2}}\right)^n\right).$$

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One more representation:

$$M_n(\cos u) = \frac{1}{2}\left(\left(\sqrt{\frac{\xi^2+\alpha^2}{1+\alpha^2\xi^2}}\right)^n + \left(\sqrt{\frac{1+\alpha^2\xi^2}{\xi^2+\alpha^2}}\right)^n,\right),$$

$$\xi = e^{iu}, \quad \alpha = \frac{\sqrt{1+a^2}-1}{a}, \quad u \in [0, \pi].$$

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2. Chebyshev – Markov rational fractions. Theorem 1

The system of Chebyshev – Markov rational fractions $M_n(x)$, $n = 0, 1, \ldots,$ *is orthogonal on the segment* $[-1, 1]$ *with respect to the weight*

$$\rho(x, a) = \frac{\sqrt{1+a^2}}{(1+a^2x^2)\sqrt{1-x^2}}, \quad -1 < x < 1, \tag{4}$$

i.e.

$$\int_{-1}^{1} M_n(x) M_m(x) \rho(x, a)\, dx = \begin{cases} 0, & m \neq n, \ m, n = 0, 1, \ldots, \\ \pi/2, & m = n = 1, 2, \ldots, \\ \pi, & m = n = 0. \end{cases}$$

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3. Fourier series and Dirichlet integral

$f(x)$:

$$\int_{-1}^{1} |f(x)|\rho(x,a)\,dx < \infty.$$

Fourier series:

$$f(x) \sim \frac{c_0}{2} + \sum_{n=1}^{+\infty} c_n M_n(x), \tag{5}$$

$$c_n = \frac{2}{\pi}\int_{-1}^{1} f(x) M_n(x)\rho(x,a)\,dx, \quad n = 0, 1, \ldots . \tag{6}$$

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3. Fourier series and Dirichlet integral. Theorem 2

For the partial sums of the Fourier series with respect to the system (3) *the following formula holds:*

$$S_n(x,f)=\frac{1-\alpha^4}{2\pi}\int\limits_{-\pi}^{\pi} f(\cos\nu)\frac{\sin\left(n+\frac{1}{2}\right)\lambda(u,\nu)}{\sin\frac{\lambda(u,\nu)}{2}}\frac{d\nu}{1+2\alpha^2\cos 2\nu+\alpha^4}, \tag{7}$$

where

$$\lambda(u,\nu)=\int\limits_{u}^{\nu}\frac{1-\alpha^4}{1+2\alpha^2\cos 2y+\alpha^4}dy,\quad \alpha=\frac{\sqrt{1+a^2}-1}{a}.$$

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3. Fourier series and Dirichlet integral. Theorem 2

Besides,

$$S_n(x, f) = \frac{p_n(x)}{\sqrt{(1 + a^2x^2)^n}},$$

where $p_n(x)$ *is an algebraic polynomial of degree at most* n *and* $S_n(x;1) \equiv 1$*.*

3. Fourier series and Dirichlet integral. Theorem 3

The following representation of the partial sum of the Fourier series with respect to the system (3) *holds:*

$$S_n(x; f) = \frac{1}{\pi}\int\limits_0^{\pi/2}\left(f\left(\frac{\cos(\theta+2\tau)}{\sqrt{1+a^2\sin^2(\theta+2\tau)}}\right)+\right.$$

$$\left.+f\left(\frac{\cos(\theta-2\tau)}{\sqrt{1+a^2\sin^2(\theta-2\tau)}}\right)\right)\frac{\sin(2n+1)\tau}{\sin\tau}\,d\tau, \quad (8)$$

where

$$\theta = \arccos x\sqrt{\frac{1+a^2}{1+a^2x^2}}. \quad (9)$$

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4. Fourier decomposition of the function $|x|$. Theorem 4

The coefficients of the decomposition

$$|x| = \frac{c_0}{2} + \sum_{m=1}^{+\infty} c_{2m} M_{2m}(x), \quad x \in [-1, 1], \tag{10}$$

can be found by following formulas:

$$c_0 = \frac{4}{\pi\alpha} \ln\left(\alpha + \sqrt{1+\alpha^2}\right), \tag{11}$$

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$$c_{2m} = \frac{2}{\pi}\frac{1-\alpha^4}{\alpha^{2m}}\left[\frac{1+\alpha^2}{4\alpha}\ln\frac{1+\alpha}{1-\alpha} - \frac{1-(-1)^m\alpha^{2m}}{2m(1+\alpha^2)}\right.$$

$$+\sum_{j=1}^{m-1}\frac{(-1)^j C_{m-1}^j(1-\alpha^4)^j}{2j+1}\left(1-(1-\alpha^2)\frac{2j+1}{2j+2}\right)\sum_{\nu=1}^{j}\frac{A_{\nu j}}{(1-\alpha^2)^{j+1-\nu}}$$

$$+\frac{1}{\alpha}\ln\frac{1+\alpha}{1-\alpha}$$

$$\left.\times\sum_{j=1}^{m-1}(-1)^j C_{m-1}^j(1-\alpha^4)^j\left(1-(1-\alpha^2)\frac{2j+1}{2j+2}\right)\frac{(2j-1)!!}{2^{j+1}j!}\right], \qquad (12)$$

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$$A_{\nu j} = \frac{(2j+1)(2j-1)(2j-3)\dots(2j-2\nu+3)}{2^{\nu} j(j-1)\dots(j+1-\nu)},$$

$$\nu = 1, 2, \dots, j, \; j = 1, 2, \dots, m-1, \quad m = 1, 2, \dots .$$

4. Fourier decomposition of the function $|x|$. Theorem 4. Proof

(11) — direct integration.

(12):

$$c_{2m} = \frac{2}{\pi}(-1)^m(1-\alpha^2)\sqrt{1+\alpha^2}\int_0^1\left(\frac{r^2-\alpha^2}{1-\alpha^2 r^2}\right)^{m-1}\frac{(r^2-1)\,dr}{(1-\alpha^2 r^2)^2},$$

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5. Estimation of approximation error of the function $|x|$

Partial sum:

$$s_{2n}(x, |x|) = s_{2n}(x) = \frac{c_0}{2} + \sum_{k=1}^{n} c_{2k} M_{2k}(x), \quad x \in [-1, 1]. \tag{13}$$

Errors:

$$\begin{gathered} \varepsilon_{2n}(x, \alpha) = |x| - s_{2n}(x), \quad x \in [-1, 1], \\ \varepsilon_{2n}(\alpha) = \big\| |x| - s_{2n}(x) \big\|_{C[-1,1]}. \end{gathered} \tag{14}$$

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5. Estimation of approximation error of the function $|x|$. Theorem 5

If $x = \cos u$, $x \in [-1, 1]$, *then:*

$$|\varepsilon_{2n}(x, \alpha)| \leq \frac{2}{\pi} \int_0^1 \sqrt{\frac{1 + 2\alpha^2 \cos 2u + \alpha^4}{1 + 2t^2 \cos 2u + t^4}} \frac{1 - t^2}{1 - t^2\alpha^2} |\chi_{2n}(t)|\, dt, \tag{15}$$

$$\varepsilon_{2n}(\alpha) \leq \frac{4}{\pi} \int_0^1 |\chi_{2n}(t)| \frac{dt}{1 + t^2}, \quad n = 0, 1, \ldots, \tag{16}$$

where $\chi_{2n}(t) = \left(\frac{t^2 - \alpha^2}{1 - \alpha^2 t^2} \right)^n$.

If all the poles have even multiplicity then (15) *becomes equality for* $x = 0$ *and* $x = \pm 1$.

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$\alpha = 0$:

$$|\varepsilon_{2n}(x)| \leq \frac{2}{\pi} \int\limits_0^1 \frac{1 - t^2}{\sqrt{1 + 2t^2 \cos 2\nu + t^4}} t^{2n}\, dt, \quad x \in [-1, 1],$$

$$\|\varepsilon_{2n}(x)\|_{C[-1,1]} = \frac{2}{\pi} \int\limits_0^1 t^{2n}\, dt = \frac{2}{\pi(2n+1)}, \quad n = 0, 1, \ldots .$$

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The following relations hold:

1. $\overline{\lim_{n\to\infty}} \dfrac{n^2}{\ln n}\varepsilon_{2n} = \dfrac{1}{\pi}$,
2. *If* n *is even, then* $\lim_{n\to\infty} \dfrac{n^2}{\ln n}\varepsilon_{2n} = \dfrac{1}{\pi}$,
3. $\inf_{\alpha} |\varepsilon_{2n}(x,\alpha)| \leq \dfrac{2}{\pi|x|}\dfrac{\ln n}{n^3}, \quad x \in [-1,0) \cup (0,1], \; n > n_0.$

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4. Evgrafov M.A. *Asymptotic estimates and entire functions.* Gordon and Breach, Moscow, 1961.
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