Some properties of Marcinkiewicz means with respect to Walsh system

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Let denote by

\[ \mathbb{Z}_2 \]

the discrete cyclic group of order 2, that is \( \mathbb{Z}_2 = \{0, 1\} \),
the group operation is the modulo 2 addition, every subset is open.
Haar measure on \( \mathbb{Z}_2 \) is given in the way that the measure of a singleton is 1/2.

The Walsh group:

\[ G := \prod_{k=0}^{\infty} \mathbb{Z}_2. \]

The elements of \( G \) are of the form

\[ x = (x_0, x_1, ..., x_k, ...) \]

with \( x_k \in \{0, 1\} \) (\( k \in \mathbb{N} \)).
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with \( x_k \in \{0, 1\} \ (k \in \mathbb{N}) \).
The **group operation** on $G$ is the coordinate-wise addition, the measure (denoted by $\mu$) and the topology are the product measure and topology.

**Fine’s map:**
For $x \in G$ we define $|x|$ by $|x| := \sum_{j=0}^{\infty} x_j 2^{-j-1}$. 
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For $x \in G$ we define $|x|$ by $|x| := \sum_{j=0}^{\infty} x_j 2^{-j-1}$. 
Rademacher functions:

\[ r_k(x) := (-1)^{x_k} \quad (x \in G, k \in \mathbb{N}). \]

If \( n \in \mathbb{N} \), then

\[ n = \sum_{i=0}^{\infty} n_i 2^i, \quad n_i \in \{0, 1\} \quad (i \in \mathbb{N}). \]

Let be the order of \( n \)

\[ |n| := \max\{j \in \mathbb{N} : n_j \neq 0\}. \]

Walsh-Paley functions:

\[ w_n(x) := \prod_{k=0}^{\infty} (r_k(x))^{n_k} = r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_k} \]

Walsh-Paley system: \( (w_n : n \in \mathbb{N}) \)
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Walsh-Paley system: \((w_n : n \in \mathbb{N})\)
Walsh-Kaczmarz functions:

\[
\kappa_n(x) := r_{|n|}(x) \prod_{k=0}^{\left|n\right|-1} (r_{|n|-1-k}(x))^{n_k}
\]

\[
= r_{|n|}(x) \sum_{k=0}^{\left|n\right|-1} n_k x^{\left|n\right|-1-k},
\]

The Walsh-Kaczmarz system:

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\kappa := (\kappa_n : n \in \mathbb{N}).
\]

It is well known that

\[
\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{w_n : 2^k \leq n < 2^{k+1}\}
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for all \(k \in \mathbb{N}\) and \(\kappa_0 = w_0\).
### Walsh-Kaczmarz functions:

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\kappa_n(x) := r_{\lfloor n \rfloor}(x) \prod_{k=0}^{\lfloor n \rfloor - 1} (r_{\lfloor n \rfloor - 1-k}(x))^{n_k}
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### The Walsh-Kaczmarz system:

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A relation between Walsh-Kaczmarz functions and Walsh-Paley functions:

The transformation \( \tau_A : G \rightarrow G \) \((A \in \mathbb{N})\). given by V. A. Skvortsov

\[
\tau_A(x) := (x_{A-1}, x_{A-2}, \ldots, x_1, x_0, x_A, x_{A+1}, \ldots)
\]

The relation

\[
\kappa_n(x) = r_{|n|}(x)w_{n-2|n|}(\tau_{|n|}(x)) \quad (n \in \mathbb{N}, x \in G).
\]
Fourier coefficients, partial sums, Dirichlet kernels, Fejér means, Fejér kernels:

\[
\hat{f}_\psi(n) := \int_G f \psi_n, \quad S_n^\psi f := \sum_{k=0}^{n-1} \hat{f}_\psi(k) \psi_k, \\
D_n^\psi := \sum_{k=0}^{n-1} \psi_k, \quad \sigma_n^\psi f := \frac{1}{n} \sum_{k=0}^{n-1} S_k^\psi f, \\
K_n^\psi := \frac{1}{n} \sum_{k=0}^{n-1} D_k^\psi,
\]

where \( \psi = \psi \) or \( \kappa \).
Let $L_p$ denote the usual Lebesgue space with the norm (or quasinorm) $\| \cdot \|_p \quad (0 < p < \infty)$.

The space weak-$L_p$ consists of all measurable function $f$ for which

$$\| f \|_{\text{weak-}L_p} := \sup_{\lambda > 0} \lambda \mu(\{|f| > \lambda\}^{1/p} < \infty.$$

Let the operator $T: H_p \to L_p$. The operator $T$ is of type $(H_p, L_p)$ if there exists a constant $c_p > 0$ such that

$$\| Tf \|_p \leq c_p \| f \|_{H_p} \quad \text{for all } f \in H_p.$$
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Introduction

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Two-dimensional systems: The Kronecker product $(\psi_{n,m} : n, m \in \mathbb{N})$ of two Walsh (-Kaczmarz) system, where
\[
\psi_{n,m}(x^1, x^2) = \psi_n(x^1) \psi_m(x^2),
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where $\psi = w$ or $\kappa$.

Two-dimensional Walsh-(Kaczmarz-)Fourier coefficient:
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\hat{f}_\psi(n, m) := \int_{G^2} f \psi_{n,m} \quad (n, m \in \mathbb{N})
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Rectangular partial sum of the Walsh-Fourier series, the Marcinkiewicz-Fejér means
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S^n_\psi(f; x^1, x^2) := \sum_{k=0}^{n-1} \sum_{i=0}^{m-1} \hat{f}_\psi(k, i) \psi_{k,i}(x^1, x^2).
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\[
M^n_\psi(f; x^1, x^2) := \frac{1}{n} \sum_{k=0}^{n-1} S^n_\psi(f; x^1, x^2).
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Definitions and notations

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- I. Marcinkiewicz (1939) for $f \in L \log L([0, 2\pi]^2)$ and for trigonometric system the mean

$$M_n(f) = \frac{1}{n} \sum_{k=0}^{n-1} S_{k,k}(f)$$

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Historical notes on the Walsh-(Kaczmarz-)Marcinkiewicz means


For $f$ we consider the maximal operator

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\mathcal{M}^\psi f(x^1, x^2) = \sup_{n \in \mathbb{P}} |\mathcal{M}_n^\psi(f; x^1, x^2)|.
$$

Connecting results: the maximal operator $\mathcal{M}^*$ is of weak type $(1, 1)$ and of type $(p, p)$ for all $1 < p \leq \infty$. 

Some properties of ...
Historical notes on the Walsh-(Kaczmarz-)Marcinkiewicz means


For $f$ we consider the maximal operator

$$\mathcal{M}^{\psi,*}f(x^1, x^2) = \sup_{n \in \mathbb{P}} |\mathcal{M}_n^{\psi}(f; x^1, x^2)|.$$  

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Historical notes on the Walsh-(Kaczmarz-)Marcinkiewicz means

- F. Weisz (2001) The maximal operator $\mathcal{M}^{w,*}$ is bounded from the two-dimensional dyadic martingale Hardy space $H_p$ to the space $L_p$ for $1 \geq p > 2/3$. Appr. Theory Appl. 17 (2001) 32-44.


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What does happen in the end point case $p = 2/3$?

**Direction 1:**

- **U. Goginava (2008):** for Walsh-Paley system, there exists a martingale $f \in H_{2/3}$ such that

  $$\|\mathcal{M}^{w,*}f\|_{2/3} = +\infty.$$


- **U. Goginava and K. Nagy (2009):** for Walsh-Kaczmarz system, analogous result

What does happen in the end point case $p = 2/3$?

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- **U. Goginava (2008):** for Walsh-Paley system There exists a martingale $f \in H_{2/3}$ such that

\[ \|M_{w,*}^w f\|_{2/3} = +\infty. \]


Direction 2:
Define the maximal operator $\tilde{M}^*$ by

$$\tilde{M}^* f := \sup_{n \in P} \frac{|M_n f|}{\log^{3/2}(n + 1)}.$$ 

- K. Nagy (2011) The maximal operator $\tilde{M}^{w,*}$ is bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$. Moreover, the following holds:
Let $\varphi : \mathbb{P} \to [1, \infty)$ be a non-decreasing function satisfying the condition

$$\lim_{n \to \infty} \frac{\log^{3/2}(n + 1)}{\varphi(n)} = +\infty$$

Then the maximal operator $\sup_{n \in \mathbb{P}} \frac{|M_n f|}{\varphi(n)}$ is not bounded from the Hardy space $H_{2/3}$ to the space $L_{2/3}$.

The order of the deviant behaviour of the $n$th Walsh-Marcinkiewicz means is

$$\log^{3/2}(n + 1).$$

-K. Nagy (2015): for Walsh-Kaczmarz system analogical result

Direction 3:

Theorem (U. Goginava, K. Nagy (2016))

The maximal operator $M^{κ,*}$ is bounded from the Hardy space $H_{2/3}$ to the space weak-$L_{2/3}$.

Direction 4:
- K. Nagy, G. Tephnadze (2014): a necessary and sufficient condition for the convergence of Walsh-Marcinkiewicz means in terms of the modulus of continuity on the Hardy space $H_{2/3}(G^2)$.


Let us define the modulus of continuity in the Hardy space $H_p$ by

$$\omega \left( \frac{1}{2^n}, f \right)_{H_p} := \| f - S_{2^n, 2^n}(f) \|_{H_p}$$
Theorem

a) Let

$$\omega \left( \frac{1}{2^k}, f \right)_{H^{2/3}} = o \left( \frac{1}{k^{3/2}} \right),$$

as \( k \to \infty \). Then

$$\|M_n(f) - f\|_{H^{2/3}} \to 0, \text{ when } n \to \infty.$$

b) There exists a martingale \( f \in H^{2/3} \), for which

$$\omega \left( \frac{1}{2^{2^k}}, f \right)_{H^{2/3}} = O \left( \frac{1}{2^{3k/2}} \right),$$

as \( k \to \infty \) and

$$\|M_n(f) - f\|_{2/3} \nrightarrow 0 \text{ as } n \to \infty.$$
Direction 5:

**Theorem (K. Nagy, G. Tephnadze (2016))**

There exists an absolute constant $c$, such that

$$\frac{1}{\log n} \sum_{m=1}^{n} \frac{\|M_w^m(f)\|_{H^{2/3}}^{2/3}}{m} \leq c \|f\|_{H^{2/3}}^{2/3},$$

for all $f \in H^{2/3}(G^2)$.


for Walsh-Kaczmarz system it is open problem.
What does happen in the case $0 < p < 2/3$?

Define the maximal operator $\tilde{\sigma}^*,p$ by

$$\tilde{\mathcal{M}}^*,p (f) := \sup_{n \geq 1} \left| \frac{\mathcal{M}_n(f)}{n^{2/p-3}} \right|,$$

**Theorem (K. Nagy, G. Tephnadze )**

*a) Let $0 < p < 2/3$. Then the maximal operator $\tilde{\mathcal{M}}^*,p$ is bounded from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$.*

*b) Let $\varphi: \mathbb{N} \to [1, \infty)$ be a non-decreasing function, satisfying the condition

$$\lim_{n \to \infty} \frac{n^{2/p-3}}{\varphi(n)} = +\infty. \quad (1)$$

Then

$$\sup_{n \in \mathbb{N}} \left\| \frac{\mathcal{M}_n f}{\varphi(n)} \right\|_{\text{weak-}L_p} = \infty.$$
New results in the case $0 < p < 2/3$

What does happen in the case $0 < p < 2/3$?
Define the maximal operator $\tilde{\sigma}^{\ast,p}$ by

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a) Let $0 < p < 2/3$. Then the maximal operator $\tilde{\mathcal{M}}^{\ast,p}$ is bounded from the Hardy space $H_p(G^2)$ to the space $L_p(G^2)$.

b) Let $\varphi : \mathbb{N} \to [1, \infty)$ be a non-decreasing function, satisfying the condition

$$\lim_{n \to \infty} \frac{n^{2/p-3}}{\varphi(n)} = +\infty. \quad (1)$$

Then

$$\sup_{n \in \mathbb{N}} \left\| \frac{\mathcal{M}_n f}{\varphi(n)} \right\|_{\text{weak-}L_p} = \infty.$$
That is, the exact order of deviant behaviour of the $n$-th Walsh-Marcikiewicz mean is calculated in Hardy space $H_p$ for $0 < p < 2/3$. It is

$$n^{2/p-3}.$$ 


After this two applications were given.

**Application 1:** A necessary and sufficient condition for the convergence of Walsh-Marcinkiewicz means in terms of the modulus of continuity on the Hardy space $H_p(G^2)$ for $0 < p < 2/3$.

**Application 2:** A strong convergence theorem.
New results in the case $0 < p < 2/3$

Theorem (K. Nagy, G. Tephnadze)

a) Let $1/2 < p < 2/3$, $f \in H_p(G^2)$ and

$$\omega \left( \frac{1}{2^k}, f \right)_{H_p} = o \left( \frac{1}{2^{k(2/p-3)}} \right),$$

as $k \to \infty$. Then

$$\| M_n(f) - f \|_{H_p} \to 0, \text{ when } n \to \infty.$$

b) Let $0 < p < 2/3$. Then there exists a martingale $f \in H_p(G^2)$, such that

$$\omega \left( \frac{1}{2^k}, f \right)_{H_p} = O \left( \frac{1}{2^{k(2/p-3)}} \right),$$

as $k \to \infty$ and

$$\| M_n(f) - f \|_{weak-L_p} \not\to 0 \text{ as } n \to \infty.$$
Theorem (K. Nagy, G. Tephnadze)

a) Let $0 < p < 2/3$. Then there exists an absolute constant $c_p$, such that

$$
\sum_{m=1}^{\infty} \frac{\|M_m f\|^p_{H_p}}{m^{3-3p}} \leq c_p \|f\|^p_{H_p}
$$

for all $f \in H_p(G^2)$.

b) Let $0 < p < 2/3$ and $\Phi: \mathbb{N}_+ \to [1, \infty)$ be any non-decreasing function, satisfying the conditions $\Phi(n) \uparrow \infty$ and

$$
\lim_{k \to \infty} \frac{2^{k(3-3p)}}{\Phi(2^k)} = \infty.
$$

Then there exists a martingale $f \in H_p(G^2)$, such that

$$
\sum_{m=1}^{\infty} \frac{\|M_m f\|^p_{weak-L_p}}{\Phi(m)} = \infty.
$$
Thanks for your attention!