STEINHAUS TILING SETS

Mihalis Kolountzakis

University of Crete

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Joint work with M. Papadimitrakis
The classical Steinhaus question

> Steinhaus (1950s): Are there \( A, B \subseteq \mathbb{R}^2 \) such that

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|\tau A \cap B| = 1, \text{ for every rigid motion } \tau.
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Are there two subsets of the plane which, no matter how moved, always intersect at exactly one point?
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- In tiling language:

$$\rho A \oplus B = \mathbb{R}^2, \quad \text{for all rotations } \rho.$$  

Every rotation of $A$ tiles (partitions) the plane when translated at the locations $B$.  

Fixing \( B = \mathbb{Z}^2 \): the lattice Steinhaus question

- Can we have \( \rho A \oplus \mathbb{Z}^2 = \mathbb{R}^2 \) for all rotations \( \rho \)?

- Equivalent: \( A \) is a fundamental domain of all \( \rho \mathbb{Z}^2 \). Or, \( A \) tiles the plane by translations at any \( \rho \mathbb{Z}^2 \).
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- Jackson and Mauldin, 2002: Yes.

- Can $A$ be Lebesgue measurable? We interpret tiling almost everywhere.
  - If such a measurable $A$ exists then it must be large at infinity: 
    \[ \int_{\mathbb{R}^2} |x|^{46/27 + \epsilon} \, dx = \infty. \]
  - No measurable Steinhaus sets exist for $\mathbb{Z}^d$, $d \geq 3$. 


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\begin{itemize}
    \item Can we have \( \rho A \oplus \mathbb{Z}^2 = \mathbb{R}^2 \) for all rotations \( \rho \)?
    \item Equivalent: \( A \) is a fundamental domain of all \( \rho \mathbb{Z}^2 \).
    Or, \( A \) tiles the plane by translations at any \( \rho \mathbb{Z}^2 \).
    \item Jackson and Mauldin, 2002: Yes.
    \item Can \( A \) be Lebesgue measurable? We interpret tiling almost everywhere. Results by Sierpiński (1958), Croft (1982), Beck (1989), K. & Wolff (1999):
    If such a measurable \( A \) exists then it must be large at infinity:
    \[
    \int_{A} |x|^{\frac{46}{27} + \epsilon} \, dx = \infty.
    \]
    \item In higher dimension:
    \( \implies \) No measurable Steinhaus sets exist for \( \mathbb{Z}^d \), \( d \geq 3 \).
\end{itemize}
The lattice Steinhaus question in Fourier space

- For $f$ to tile with $\mathbb{Z}^2$ its periodization

$$\sum_{n \in \mathbb{Z}^2} f(x - n)$$

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  must be constant.
- Equivalently $\hat{f}(n) = 0$ for all $n \in \mathbb{Z}^2 \setminus \{0\}$.
- Applying to $f = 1_{\rho A}$ for all rotations $\rho$ we get that $\hat{1}_A$ must vanish on all circles through lattice points.
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The lattice Steinhaus question for finitely many lattices

Given lattices $\Lambda_1, \ldots, \Lambda_n \subseteq \mathbb{R}^d$ all of volume 1 can we find measurable $A$ which tiles with all $\Lambda_j$?
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Generically yes!
If the sum \( \Lambda_1^* + \cdots + \Lambda_n^* \) is direct then Kronecker-type density theorems allow us to rearrange a fundamental domain of one lattice to accommodate the others.
If $G$ is an abelian group and $H_1, \ldots, H_n$ subgroups of same index.

- Always possible for two subgroups $H_1, H_2$ (even in non-abelian case).
- Fails in general: take $G = \mathbb{Z}_2 \times \mathbb{Z}_2$ and the 3 copies of $\mathbb{Z}_2$ therein.
- No good condition is known!
If $G$ is an abelian group and $H_1, \ldots, H_n$ subgroups of same index, can we find a common set of coset representatives for the $H_j$?
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Question: If $K, L$ are two lattices in $\mathbb{R}^d$ with

$$\text{vol } K \cdot \text{vol } L = 1,$$

can we find $g \in L^2(\mathbb{R}^d)$, such that the $(K, L)$ time-frequency translates

$$g(x - k)e^{2\pi i \ell \cdot x}, \quad (k \in K, \ell \in L)$$

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The space is partitioned in copies of $E$ and on each copy $L$ is an orthogonal basis.
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- $B$ is a finite set:

The shaded set tiles with $B$
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- Komjáth (1992): There is \( A \subseteq \mathbb{R}^2 \) such that for all rotations \( \rho \)

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\rho A \oplus (\mathbb{Z} \times \{0\}) \text{ is a tiling.}
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- For $B \subseteq \mathbb{R}^2$ finite and of size 3, 4, 5, 7:

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  \[ \implies \text{No such sets } A. \]

WE SHOW HERE

- A Komjáth set cannot be Lebesgue measurable.
- For any finite \( B \subseteq \mathbb{R}^2 \) there is no Lebesgue measurable Steinhaus set \( A \).
\textbf{Finite \textit{B}: a Fourier condition}

Write $\delta_B = \sum_{b \in B} \delta_b$.

$\Rightarrow \hat{\delta_B}(x) = \sum_{b \in B} e^{-2\pi i b \cdot x}$ is a trig. polynomial.

\begin{itemize}
  \item Suppose $1_A \ast \delta_B(x) = \sum_{b \in B} 1_A(x - b) = 1 \text{ a.e.}(x)$
\end{itemize}
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- Suppose $1_A * \delta_B(x) = \sum_{b \in B} 1_A(x - b) = 1$ a.e.(x)
- Taking Fourier Transform:

  $$\widehat{1_A} \cdot \widehat{\delta_B} = \delta_0.$$
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- We conclude
  $$\text{supp} \hat{1_A} \subseteq \{0\} \cup \{\hat{\delta_B} = 0\}.$$  

(Notice $\hat{1_A}$ is a *tempered distribution*.)
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(Notice $\hat{1}_A$ is a \textit{tempered distribution}.)

- Valid for all rotations $\rho$:

$$\bigcup_{\rho} \rho \left( \text{supp } \hat{1}_A \right) \subseteq \{0\} \cup \{\hat{\delta}_B = 0\}.$$  

$\implies$ The zeros of $\hat{\delta}_B$ contain a \textit{circle}.
Zeros of trigonometric polynomials

**Theorem**

If \( \psi(x) = \sum_{b \in B} c_b e^{2\pi i b \cdot x} \) is a trigonometric polynomial on \( \mathbb{R}^d \) which vanishes on a sphere then \( \psi(x) \equiv 0 \).
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- Enough to prove for \( d = 2 \). May assume zeros at unit circle centered at origin.
- May also assume \( (b_0, 0) \in B \) is unique with maximal modulus.
Write $b = b_x + ib_y$, for $b \in B$, and $z = x - iy$, with $|z| = 1$. Then $(b_x, b_y) \cdot (x, y) = \Re(bz)$ and

$$\psi(x, y) \sum_{b \in B} c_b e^{2\pi i \Re(bz)} |z|=1 \sum_{b \in B} c_b e^{\pi i (bz + \frac{b}{z})} =: g(z)$$

vanishes at $|z| = 1$, hence $g(z) \equiv 0$ for all $z \neq 0$. 
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For real $t \to +\infty$ we have

$$
0 = g(-it) = c_{b_0} e^{\pi b_0 t + O(1/t)} + \sum_{b \in B \setminus \{(b_0,0)\}} c_b e^{\pi ibt + O(1/t)}
$$

Contradiction for:

unique exponential with highest exponent.
Suppose $B = \{(n, 0) : n \in \mathbb{Z}\} \subseteq \mathbb{R}^2$ and measurable $A$ so that

$$\sum_{n \in \mathbb{Z}} 1_{\rho A}(x - n, y) = 1,$$

for all rotations $\rho$ and a.e. $(x, y)$. 

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$\implies A$ has infinite measure.
Komjáth sets

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- $\implies A$ has infinite measure.

- Integrating for $x \in [0, 1]$ gives that

$$\rho A \cap (\mathbb{R} \times \{y\})$$

has measure 1 for almost all $y \in \mathbb{R}$.

- Hence $A$ intersects almost all lines of the plane at measure 1.
Komjáth sets: meeting the lines thus is too much

**Theorem**

*There is no measurable* $A \subseteq \mathbb{R}^2$ *which intersects almost all lines of the plane in measure (length) at least* $C_1$ *and at most* $C_2$, *where* $0 < C_1, C_2 < \infty$.

- We only need $C_1 = C_2 = 1$ for showing there are no measurable Komjáth sets.
Suppose $A \subseteq \mathbb{R}^2$ has the bounded line intersection property. View $\mathbb{R}^2$ embedded in $\mathbb{R}^3$.

Define $f : \mathbb{R}^3 \to \mathbb{R}^+$ by (convergence is clear)

$$f(z) = \int_{\mathbb{R}^2} 1_A(w) \frac{1}{|z - w|} \, dw.$$
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Claim: $C_1 \pi \leq f(z) \leq C_2 \pi$ for almost all $z \in \mathbb{R}^2$

$$f(z) = \int_{\mathbb{R}^2} \mathbf{1}_A(w) \frac{dw}{|z - w|}$$

$$= \int_{\mathbb{R}^2} \mathbf{1}_A(z + w) \frac{dw}{|w|} \quad \text{(change of variable)}$$

$$= \int_{[0, \pi]} \int_{\mathbb{R}} \mathbf{1}_A(z + r(\cos \theta, \sin \theta)) \, dr \, d\theta \quad \text{(polar coordinates)}$$

$$= \int_{[0, \pi]} |A \cap (z + L_\theta)| \, d\theta \quad \text{(where } L_\theta \text{ is the line with angle } \theta)$$

$$\in [C_1 \pi, C_2 \pi].$$
Line integrals bounded above and below, continued

- $f$ is continuous on $\mathbb{R}^3$.
  Technical proof omitted.

- $f$ is harmonic in the upper half-space $H = \{(x_1, x_2, x_3) : x_3 > 0\}$.
- Essentially because $\frac{1}{|x|}$ is harmonic in $\mathbb{R}^3 \setminus \{0\}$.

- If $z'$ is the projection of $z \in \mathbb{R}^3$ onto $\mathbb{R}^2$ then $0 \leq f(z) \leq f(z') \leq C_2 \pi$.

- Harmonic in $H$, bounded and continuous in $H = \Rightarrow$ is the Poisson mean of $f|_{\mathbb{R}^2} = \Rightarrow C_1 \pi \leq f(z) \leq C_2 \pi$ for $z \in H$.

- Contradiction: Clearly $\lim_{t \to +\infty} f(x, y, t) = 0$. 
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The end.

Thank you.