Construction of Orthogonal and Biorthogonal Product Systems

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Rademacher and Walsh Systems

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- The Rademacher functions r_n $(n \in \mathbb{N})$ can be derived from the basic function r by dilation:

$$r_n(x) := r(2^n x) \quad (x \in [0,1), n \in \mathbb{N})$$

$$r(x) := \begin{cases} 1, & x \in [k,\ k+\frac{1}{2}),\ k \in \mathbb{Z}, \\ -1, & x \in [k+\frac{1}{2},\ k+1),\ k \in \mathbb{Z}. \end{cases}$$

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- The Walsh system is the product system of the Rademacher system i.e.

$$w_m = \prod_{k=0}^{\infty} r_k^{m_k},$$

where

$$m = \sum_{k=0}^{\infty} m_k \cdot 2^k, \quad m_k \in \{0,\ 1\}.$$

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The w_m $(m \in \mathbb{N})$ Walsh system is a complete orthonormal system with respect to the scalar product

$$\langle f, g \rangle = \int_0^1 f(t) \cdot \overline{g}(t) dt.$$

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$$h_m(x) = h_{n,k}(x) = 2^{\frac{n}{2}} h(2^n x - k) = \begin{cases} 2^{\frac{n}{2}} & x \in [\frac{k}{2^n}, \frac{2k+1}{2^{n+1}}) \\ -2^{\frac{n}{2}} & x \in [\frac{2k+1}{2^{n+1}}, \frac{k+1}{2^n}) \\ 0 & \text{otherwise,} \end{cases}$$

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- The Haar system was the first and the simplest wavelet.
- Wavelet construction

Haar-functions can be derived from the basic function

$$h(x) := h_1(x) = \begin{cases} 1, & (0 \le x < 1/2), \\ -1, & (1/2 \le x < 1), \\ 0, & (1 \le x < \infty). \end{cases}$$

by translation and dilation: $h_{n,k}(x) := 2^{n/2} h(2^n x - k)$ $(x \in \mathbb{R},\ 0 \le k < 2^n, n \in \mathbb{N})$.

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- For the Haar functions and for the Haar scaling functions the following are true

$$\begin{aligned} \chi_{n,k} &= \chi_{n+1,2k} + \chi_{n+1,2k+1}, \quad (0 \le k < 2^n, n \in \mathbb{N}), \\ h_{n,k} &= 2^{n/2}(\chi_{n+1,2k} - \chi_{n+1,2k+1}) \quad (0 \le k < 2^n, n \in \mathbb{N}). \end{aligned}$$

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- The 2^n-th Dirichlet kernel of the Walsh-system is of the form

$$D_{2^n}(x, y) = \sum_{k=0}^{2^n-1} w_k(x) \cdot \overline{w}_k(y) = \prod_{j=0}^{n-1}(1 + r_j(x) r_j(y)) \quad (x, y \in [0, 1), n \in \mathbb{N})$$

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- The Haar-functions and the Haar scaling functions can be expressed as

$$\begin{aligned} h_{n,k}(x) &= 2^{-n/2} r_n(x) D_{2^n}(x, k2^{-n}) \\ \chi_{n,k}(x) &= 2^{-n} D_{2^n}(x, k2^{-n}) \quad (x \in [0,1), 0 \le k < 2^n, n \in \mathbb{N}). \end{aligned}$$

Generalization of Product system

- Let us fix the number $p \in \mathbb{N}^{**} := \{2, 3, \cdots\}$ and the non-empty set X. Let us start from the following finite collection of the systems

$$\phi_n = (\varphi_n^{(i)}, 0 \leq i < p), \quad (0 \leq n < N \leq \infty, \quad \varphi_n^{(i)} : X \to \mathbb{C})$$

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  is the **generalized product system** of the systems ϕ.
- It is really the generalization of idea of the product system, because in special case when $p = 2$ and $\varphi_m^{(0)} = 1$, $\varphi_m^{(1)} = r_m$ we reobtain the Walsh system.

Biorthogonality with respect to a p-fold map

- Let us fix the number $p \in \mathbb{N}^{**} := \{2, 3, \cdots\}$ and the set $X \neq \emptyset$. The map $A : X \to X'$ is called **p-fold map on set X** if every $x \in X'$ has exactly p preimages, i.e. the set

$$A^{-1}(x) = \{x_0, x_1, \ldots, x_{p-1}\} \quad x \in X'.$$

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- Let

$$f^{(j)}, g^{(j)} : X \to \mathbb{C}, \quad j = 0, 1, \dots, p-1.$$

  and $\rho : X \to (0, +\infty)$ is a positive weight-function. The system $F = (f^{(0)}, f^{(1)}, \dots f^{(p-1)})$ and the system $G = (g^{(0)}, g^{(1)}, \dots g^{(p-1)})$ is called (A, ρ)**-biorthogonal** if for every $x \in X'$

$$(\maltese) \qquad \sum_{t \in A^{-1}(x)} f^{(i)}(t) \cdot \overline{g^{(j)}(t)} \rho(t) = \sum_{k=0}^{p-1} f^{(i)}(x_k) \cdot \overline{g^{(j)}(x_k)} \rho(x_k) = \delta_{ij}$$

  if $F = G$ the system F is called (A, ρ)**-orthonormal**.

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- We will prove, that condition ✠ is equivalent with

$$(\clubsuit) \qquad \sum_{i=0}^{p-1} f^{(i)}(x_k) \cdot \overline{g^{(i)}(x_\ell)} \rho(x_k) = \delta_{k\ell} \quad (0 \leq k, \ell < p)$$

Proof:

Let us introduce the following matrices:

$$F := \begin{bmatrix} f^{(0)}(x_0) & f^{(0)}(x_1) & \dots & f^{(0)}(x_{p-1}) \\ f^{(1)}(x_0) & f^{(1)}(x_1) & \dots & f^{(1)}(x_{p-1}) \\ \vdots & \vdots & & \vdots \\ f^{(p-1)}(x_0) & f^{(p-1)}(x_1) & \dots & f^{(p-1)}(x_{p-1}) \end{bmatrix}$$

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Then the condition

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can be written as

$$(FR\overline{G}^T)_{ij} = \delta_{ij}$$

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written with the elements of matrices we get

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□

The Discrete Set X_n

Assume that for the elements $A_n : X \to X', n \in \mathbb{N}^*$ of the sequence of p-fold maps

$$A_n^{-1}(x) \subset X' \quad (x \in X',\ n \in \mathbb{N}^*).$$

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By composition of these maps we get

$$T_0(x) := x\ (x \in X),\ T_n = A_n \circ A_{n-1} \circ \ldots \circ A_1 = A_n \circ T_{n-1}\ (n \in \mathbb{N}^*).$$

Starting from a fixed element $x_0 \in X'$ we can define the preimage of this element in map T_n, this discrete set is denoted by

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If x_0 is fixed-point of every maps A_n, i.e. $A_n(x_0) = x_0$, $(n \in \mathbb{N})$, then

$$X_n \supset X_{n-1} \supset \ldots \supset X_1.$$

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The point x_0 has p preimages in map A_n: $A_n^{-1}(x_0) = \{x_0, x_1, \ldots, x_{p-1}\}$. The set X_n can be written in form

$$X_n = X_{n-1} \cup X_{n-1}^1 \cup X_{n-1}^2 \cup \ldots \cup X_{n-1}^{p-1}$$

where $X_{n-1}^j = T_{n-1}(x_j)$ $(j = 1, 2, \ldots, p-1)$.

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- $A^j(x_\ell^n) = x_{[\ell p^{-j}]}^{n-j} \quad (0 \leq \ell < p^n, 0 \leq j \leq n)$.

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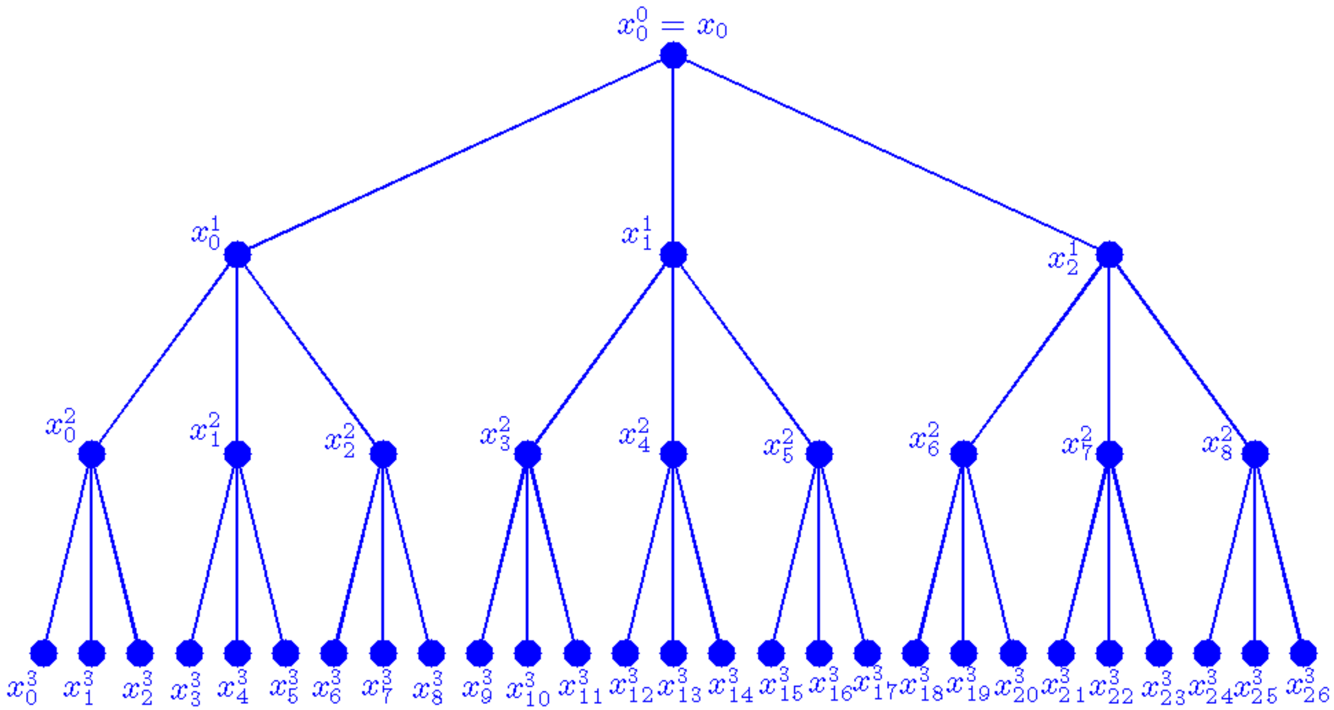
- Two of these subsets are always disjoint or one of them includes the other.

The set X_n in case $p = 3$:



The Discrete Biorthogonal Product System

- Denote $F_n := (f_n^{(0)}, f_n^{(1)}, \ldots, f_n^{(p-1)})$ and $G_n := (g_n^{(0)}, g_n^{(1)}, \ldots, g_n^{(p-1)})$ two sequences of systems and $\rho : X \to (0, +\infty) \quad (n \in \mathbb{N})$ a positive weight function, where

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$$\Phi_n := (\phi_n^{(i)}, 0 \le i < p), \ \Gamma_n := (\gamma_n^{(i)}, 0 \le i < p), \quad (n \in \mathbb{N}).$$

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- The product systems of these systems are called Walsh-like systems and can be written in form
$$\psi_m := \prod_{i=0}^{N-1} \phi_i^{m_i} \quad \eta_m := \prod_{i=0}^{N-1} \gamma_i^{m_i} \quad (0 \le m < p^N) \quad m = \sum_{i=0}^{N-1} m_i p^i,$$

Theorem 1 *The p^n-th mixed kernel of the system $(\psi_m, 0 \leq m < p^N)$ and system $(\eta_m, 0 \leq m < p^N)$ can be written in product form*

$$D_{p^n}(x,t) := \sum_{k=0}^{p^n-1} \psi_k(x)\overline{\eta_k(t)} = \prod_{i=0}^{n-1}(\phi_i^{(0)}(x)\overline{\gamma}_i^{(0)}(t) + \phi_i^{(1)}(x)\overline{\gamma}_i^{(1)}(t) + \ldots + \phi_i^{(p-1)}(x)\overline{\gamma}_i^{(p-1)}(t)),$$

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This theorem can be proven by induction for n.

Theorem 2 *The product system*

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are biorthogonal with respect to the discrete scalarproduct

$$\langle f, g \rangle := \sum_{x \in X_N} f(x)\overline{g(x)}\sigma_N(x),$$

and

$$(\star) \qquad \sigma_N(x) D_{p^N}(x,t) = \delta_{x,t} \quad (x, t \in X),$$

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Consequence 1 For any function f the p^N-th partial sum of Fourier series is

$$(S_{p^N}f)(x) = \sum_{k=0}^{p^N-1} \langle f, \eta_k \rangle \psi_k(x) = \sum_{t\in X_N} f(t) \sum_{k=0}^{p^N-1} \psi_k(x)\overline{\eta_k(t)}\sigma_N(t) = \sum_{t\in X_N} f(t) D_{p^N}(x,t)\sigma_N(t).$$

From this $(S_{p^N}f)(x) = f(x)\ x \in X_N$.

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$$L(x,y) = \phi^{(0)}(x)\overline{\gamma^{(0)}(y)} + \phi^{(1)}(x)\overline{\gamma^{(1)}(y)} + \ldots + \phi^{(p-1)}(x)\overline{\gamma^{(p-1)}(y)} \quad (x,y \in X)$$

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- Let us introduce the following analogues of the scaling functions $\chi_{n,k}$:

$$\mathcal{I}_{n,k}(x) := p^{-n}\prod_{j=0}^{n-1} L(A^{N-1-j}(x), A^{N-1-j}(x_k^n)),$$

$$(x \in X, 0 \le k < p^n, n = 1, 2, \cdots, N).$$

- The following theorem is true and it means that similarly to Haar scaling functions $\mathcal{I}_{n,k}$ is the characteristic function of the set $I_{n,k}$.

Theorem: *The scaling functions $\mathcal{I}_{n,k}$, which were made from orthonormal functions, satisfies on the set X_N*

$$\mathcal{I}_{n,k} = \chi_{I_{n,k}} \quad (0 \le k < p^n, n = 1, 2, \ldots, N),$$

thus in the points of the set X_N the following scaling equation is true

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- We showed that all factor in the product form of function $\mathcal{I}_{n,k}$ are p if $x \in I_{n,k}$.
- The mentioned product has a zero factor in case $x \notin I_{n,k}$.

Haar-like System

- We can introduce the Haar-like functions by the following equation

$$\mathcal{H}_{n,k} := \sum_{\ell=0}^{p-1} (-1)^\ell \mathcal{I}_{n+1,pk+\ell} \quad (0 \le k < p^n, n = 0, 1, \cdots, N-1).$$

Theorem: *If $p = 2q$ $(q \in \mathbb{N})$ then the discretized system of $\mathcal{H}_{n,k}$ $(0 \le k < p^n, n = 0, 1, \cdots, N-1)$ is a discrete orthogonal Haar-like system with respect to the scalar product*

$$\langle f, g \rangle := p^{-N} \sum_{x \in X_N} f(x)\overline{g}(x),$$

exactly

$$\langle \mathcal{H}_{n,k}, \mathcal{H}_{m,\ell} \rangle = p^{-n} \cdot \delta_{n,m} \cdot \delta_{k,\ell},$$
$$(0 \le k < p^n,\ 0 \le n < N,\ 0 \le \ell < p^m,\ 0 \le m < N).$$

- The previous theorem can be proven by simply computation. For the proof we used the mentioned special properties of the generalized p-adic intervals.

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Thank You For Your Attention!