FLOWS OF MELLIN TRANSFORMS

‘periodic’ generalized primes

Introduction - Motivation

Approach to Riemann Hypothesis (RH): find f_n (or flow f_λ) of holomorphic functions s.t. $f_n \to \zeta$, uniformly. If zeros of $f_n(s)$ to left of $\mathrm{Re}(s) = \frac{1}{2}$, then RH follows.

Problem: how to choose your sequence or flow?

Restrict to Mellin transforms

$$\hat{N}_\lambda(s) = \int_0^\infty x^{-s}\, dN_\lambda(x), \qquad (\lambda \in [0,1])$$

with $N_\lambda(x) \to [x]$ as $\lambda \to 1$. Thus $\hat{N}_\lambda(s) \to \zeta(s)$.
eg. Start from 'smooth' $N_0(x)$ and 'flow' to $N_1(x) = [x]$.

$$N_0(x) = \begin{cases} x & \text{if } x \geq 1 \\ 0 & \text{otherwise} \end{cases} \dashrightarrow N_1(x) = [x]$$

$$\hat{N}_0(s) = \frac{s}{s-1} \dashrightarrow \hat{N}_1(s) = \zeta(s).$$

Relevant facts on $\zeta(s)$

1. For $\mathrm{Re}(s) > 1$,

$$\zeta(s) = \sum_{n=1}^{\infty} \frac{1}{n^s} = \int_0^{\infty} x^{-s}\, d[x].$$

2. Analytic continuation to $\mathbb{C} \setminus \{1\}$; simple pole at 1
3. ζ has finite order:

$$\mu(\sigma) = \inf\{A : \zeta(\sigma + it) \ll |t|^A\} < \infty \qquad \text{convex decreasing}$$

$\mu(\sigma) = 0$ $(\sigma \geq 1)$, $\mu(\sigma) = \frac{1}{2} - \sigma$ $(\sigma \leq 0)$
Lindelöf Hypothesis (LH): $\mu(\sigma) = \mu_0(\sigma)$ where

$$\mu_0(\sigma) = \begin{cases} \frac{1}{2} - \sigma & \text{if } \sigma < \frac{1}{2} \\ 0 & \text{if } \sigma \geq \frac{1}{2} \end{cases}$$

Equivalently, $\mu(\frac{1}{2}) = 0$. RH $\Rightarrow$ LH.

4. Functional equation: $\zeta(s) \leftrightarrow \zeta(1 - s)$

Natural properties for $N(x)$

By analogy with $[x]$ and $\zeta(s)$, $N(x)$ could satisfy:

1. $N(x) = 0$ for $x < 1$, $N(1) = 1$, and N has bounded variation in each $[a, b]$. Thus $\hat{N}(s)$ is bounded in half-planes far enough to the right.
2. *Periodicity* For $x \geq 1$, $N(x) - x$ is periodic with period 1.
3. *Generalised prime systems.* $N(x)$ forms part of a generalised prime system; i.e. $\log \hat{N}(s) = \hat{\Pi}(s)$ for some increasing $\Pi(x)$.
4. *Zeros.* $\hat{N}(s)$ has all zeros in half-planes to left of the line $\text{Re}(s) = \frac{1}{2}$; i.e. $\sup\{ \text{Re}(s) : \hat{N}(s) = 0\} < \frac{1}{2}$.

We consider systems which satisfy 1. and 2. As such, $\hat{N}(s)$ has an analytic continuation to $H_0 \setminus \{1\}$ with a simple pole at 1. In fact:

Result 1

Theorem 1 *Let N satisfy 1. and 2.; i.e. $N(x) = x - R(x)$ where R has period 1. Thus R has Fourier series*

$$R(x) \sim a_0 + \sum_{n=1}^{\infty} \Big(b_n \cos 2\pi nx + c_n \sin 2\pi nx \Big).$$

Then $\hat{N}(s)$ has an analytic continuation to $\mathbb{C} \setminus \{1\}$, simple pole at 1, residue 1. Further, $\hat{N}(s)$ is of finite order and for $\sigma < 0$

$$\hat{N}(s) = \frac{s}{s-1} + \int_0^1 x^{-s}\, dR(x) + (2\pi)^s \Gamma(1-s) \Big(\cos \frac{\pi s}{2} \sum_{n=1}^{\infty} b_n n^s - \sin \frac{\pi s}{2} \sum_{n=1}^{\infty} c_n n^s \Big).$$

Corollary 2 *Under the assumptions of Theorem 1, if $R(x) \not\equiv 0$ then $\mu(\sigma) = \frac{1}{2} - \sigma$ for $\sigma \leq 0$ (at least) and $\mu(\sigma) \geq \mu_0(\sigma)$ for all σ. In particular, $\hat{N}(s)$ has infinitely many zeros in $H_{\frac{1}{2}-\delta}$ for any $\delta > 0$.*

Thus $\sup\{\mathrm{Re}(s) : \hat{N}(s) = 0\} \geq \frac{1}{2}$; i.e. 1.,2., and 4. not compatible.

Sketch of proof of Corollary

With $s = \sigma + it$

$$\left|(2\pi)^s\Gamma(1-s)\left(\cos\frac{\pi s}{2}\sum_{n=1}^{\infty} b_n n^s - \sin\frac{\pi s}{2}\sum_{n=1}^{\infty} c_n n^s\right)\right|$$

$$\sim c|t|^{\frac{1}{2}-\sigma}\left(\left|\sum_{n=1}^{\infty}(b_n \pm ic_n)n^s\right| + O(e^{-\pi|t|})\right)$$

If $R \not\equiv 0$, then there exists (least) n_0 such that $b_{n_0} + ic_{n_0} \neq 0$. Hence the above is $\asymp |t|^{\frac{1}{2}-\sigma}$ for σ sufficiently large and negative. Then use convexity to give $\mu(\sigma) \geq \mu_0(\sigma)$.

If finitely many zeros for $\sigma > \frac{1}{2} - \delta$, then $\log \hat{N}(s)$ holomorphic here (for $|t|$ large), giving $\mu(\sigma) = 0$ for $\sigma > \frac{1}{2} - \delta$.

Remark Result extends to $N(x) - cx$ almost periodic.

Beurling (or G-prime) systems – discrete

Sequence p_n satisfying

$$1 < p_1 \leq p_2 \leq \dots \leq p_n \to \infty \qquad \textit{(g-primes)}$$

g-integers: all numbers of the form $p_1^{a_1} \dots p_k^{a_k}$ $(a_i \in \mathbb{N}_0)$.
Counting functions

$$\pi(x) = \sum_{p_n \leq x} 1 \quad \text{and} \quad N(x) = \#\{\text{g-integers} \leq x\}.$$

Also define

$$\Pi(x) = \sum_{k=1}^{\infty} \frac{\pi(x^{1/k})}{k} \text{ and } \psi(x) = \sum_{p_n^k \leq x} \log p_n = \int_0^x \log t \, d\Pi(t).$$

Then $(N * \psi)(x) \stackrel{\text{def}}{=} \int_0^x \psi(\frac{x}{t}) \, dN(t) = \int_0^x \log t \, dN(t)$.

Symbolically: $N * \Pi_L = N_L$.

G-prime systems – continuous

Given $\Pi(x)$ an increasing function on $[1, \infty)$ with $\Pi(1) = 0$, let

$$N(x) = \sum_{k=0}^{\infty} \frac{\Pi^{*k}(x)}{k!} =: \exp_* \Pi(x).$$

(Π^{*k} is the k-fold convolution of Π.) Then $N * \Pi_L = N_L$.

As such, (Π, N) is an *outer g-prime system*.

Example $N(x) = cx + 1 - c$ for $x \geq 1$ and zero otherwise ($c > 0$ constant). Then $\psi'(x) = 1 - x^{-c}$ for $x > 1$ and

$$\Pi(x) = \int_1^x \frac{1 - t^{-c}}{\log t}\, dt.$$

G-prime systems – continued

Remarks

1. Given N increasing on $[1, \infty)$, can write $N = \exp_* \Pi$ for some Π, but Π may not be increasing.
2. No mention of $\pi(x)$. If

$$\pi(x) \stackrel{\text{def}}{=} \sum_{k=1}^{\infty} \frac{\mu(k)\Pi(x^{1/k})}{k}$$

is increasing, we have a *g-prime system*. Possible that Π increases but π doesn't.
In example, π is increasing for $0 < c \leq \lambda$ but not for $c > \lambda$ (some $\lambda > 2$).

Beurling PNT: $N(x) = cx + O(\frac{x}{(\log x)^{\alpha}})$ $(\alpha > \frac{3}{2})$ implies $\Pi(x) \sim \frac{x}{\log x}$.

Which g-prime systems have $N(x) - cx$ periodic?

Discrete The g-prime system containing the usual primes except $p_1, \dots p_k$ has

$$N(x) = \sum_{\substack{n \le P \\ (n,P)=1}} \left[\frac{x-n}{P} + 1\right],$$

where $P = p_1 p_2 \dots p_k$. In this case $N(x+P) = N(x) + \varphi(P)$ where φ is Euler's function, and $N(x) - \frac{\varphi(P)}{P}x$ has period P.

Continuous $N(x) = cx + 1 - c$ $(x \ge 1)$ for $0 < c < \lambda$.

Any others?

Main Results

Theorem 3 *Let N be increasing s.t. $N(x) - cx$ is smooth and periodic ($c > 0$). Then Π is increasing if and only if $N(x) = cx + 1 - c$ for $x \geq 1$.*

Theorem 4 *Let N be increasing such that $N(x) - cx$ has period P and finitely many discontinuities in $[0, P]$ but is otherwise smooth. Suppose N determines a g-prime system. Then $P \in \mathbb{N}$ and*

$$N(x) = \sum_{\substack{n \leq P \\ (n, P) = 1}} \left[\frac{x - n}{P} + 1\right].$$

i.e. N is the integer-counting function of the g-prime system $\mathbb{P} \setminus \{p_1, \ldots, p_k\}$ where $p_1, \ldots, p_k$ are the prime divisors of P.

Thus there are no others!

Sketch of proof of Theorem 3

Suppose $R(x) = cx - N(x)$ in not constant and Π increasing.

Differentiate $N_L = N * \psi$ to give

$$\int_1^x \frac{R'(x) - R'(\frac{x}{t})}{t}\, d\psi(t) \leq A. \qquad (\dagger)$$

Choose $x = x_0$ to maximize $R'(x)$.
$\exists$ interval around x_0 s.t. $R'(x_0) - R'(x) \geq \delta$.

Periodicity + $(\dagger)$ with $x = x_0 + nP$ shows LHS $(\dagger)$ is unbounded.

Sketch of proof of Theorem 4

Write $N = N_J + N_C$ and $\Pi = \Pi_J + \Pi_C$. ('J' jump; 'C' continuous)

Note: (N_J, Π_J) is a g-prime system. Then show:

- discontinuities all at integers; i.e. $N_J(x) = \sum_{n \le x} a_n$, some a_n.
- a_n is periodic
- a_n is multiplicative
- $N_C = 0$.

For outer g-primes, $\exists$ other systems; eg.

$$N(x) = [x] + \lambda \left[\frac{x}{2}\right] + \mu \left[\frac{x}{3}\right] + \lambda\mu \left[\frac{x}{6}\right], \qquad (\lambda, \mu \in [-1, 1])$$

has $N(x) - cx$ period 6 (some c).

Corollary

In proving Theorem 4, we prove the following result on Dirichlet series with periodic coefficients.

Theorem 5
Let $\{a_n\}_{n\in\mathbb{N}}$ *be periodic,* $a_1 = 1$*, and suppose* $a_n = \exp_* b_n$ *for some* $b_n \geq 0$*. Then* a_n *is multiplicative.*

Here $*$ refers to Dirichlet convolution. Thus a_n and b_n are related by $\sum_{n=1}^{\infty} \frac{a_n}{n^s} = \exp\{\sum_{n=1}^{\infty} \frac{b_n}{n^s}\}$.

A particular flow

For $\lambda \geq 1$, let $N_\lambda(x) = x - R_\lambda(x)$, where $R_\lambda(x)$ has period 1 and is defined for $0 \leq x < 1$ by

$$R_\lambda(x) = \rho_\lambda(\zeta(1-\lambda, 1-x) - \zeta(1-\lambda)) = \frac{\rho_\lambda \Gamma(\lambda)}{2\pi i} \int_C \frac{z^{-\lambda}(e^{-xz} - 1)}{e^{-z} - 1}\, dz.$$

Fourier expansion:

$$R_\lambda(x) = -\frac{2\rho_\lambda \Gamma(\lambda)}{(2\pi)^\lambda} \left(\cos \frac{\pi\lambda}{2} \sum_{n=1}^{\infty} \frac{1 - \cos 2\pi nx}{n^\lambda} + \sin \frac{\pi\lambda}{2} \sum_{n=1}^{\infty} \frac{\sin 2\pi nx}{n^\lambda} \right)$$

which holds for all $x \in \mathbb{R}$ if $\lambda > 1$ and for $x \in \mathbb{R} \setminus \mathbb{Z}$ if $\lambda = 1$.

Thus $N_\infty(x) = x$ and $N_1(x) = [x]$.

A particular flow – properties

$\hat{N}_\lambda(s) \to \zeta(s)$ locally uniformly on $\mathbb{C} \setminus \{1\}$.

Zeros:
(i) Let $\lambda \geq \frac{3}{2}$. Then for every $\delta > 0$, $\hat{N}_\lambda(s)$ has at most finitely many zeros in $H_{\frac{1}{2}+\delta}$.
(ii) Let $1 < \lambda < \frac{3}{2}$. Then for every $\delta > 0$, $\hat{N}_\lambda(s)$ has at most finitely many zeros in $H_{2-\lambda+\delta}$. On the Lindelöf Hypothesis this can be replaced by $H_{\frac{1}{2}+\delta}$.
For $\lambda > \frac{3}{2}$, $\hat{N}_\lambda(s)$ has

$$\frac{T}{2\pi} \log \frac{T}{2\pi} - \frac{T}{2\pi} + O((\log T)^\kappa)$$

zeros in the rectangular strip $\{\sigma + it : 0 \leq \sigma \leq 1, 0 \leq t \leq T\}$, where $\kappa = \max\{0, 3 - 2\lambda\}$.

References

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Generalised prime systems with periodic integer counting function, *Acta Arith.* **152** (2012) 217-241.