Divergence of Fourier series of functions with restrictions on the fractality of their graphs

Gridnev Maxim

Krasovskii Institute of Mathematics and Mechanics, Ekaterinburg, Russia

Definition

Let $f : [a, b] \to \mathbb{R}$ be a bounded function. A modulus of fractality of function f is a function $\nu(f, \varepsilon)$ that for all $\varepsilon > 0$ gives the minimal number of squares with sides of length ε which can cover graph of function f.

Let $\mu : (0, +\infty) \to \mathbb{R}$ be a nonincreasing continuous function, $\lim\limits_{\varepsilon \to 0} \mu(\varepsilon) = +\infty$. We define

$$F^\mu := \{f \in C[a, b] : \nu(f, \varepsilon) = O(\mu(\varepsilon))\}.$$

For $\mu(\varepsilon) = \frac{1}{\varepsilon^\alpha}$ we will write F_α instead of $F^{\frac{1}{\varepsilon^\alpha}}$, where $1 \leqslant \alpha \leqslant 2$.
Remark: $F_1 \leqslant F_\alpha \leqslant F_\beta \leqslant F_2$, where $\alpha \leqslant \beta$.

Definition

Let $1 \leqslant p < +\infty$, $f : [a, b] \to \mathbb{R}$,
let $\tau = \{a = t_0 < t_1 < ... < t_{n-1} < t_n = b\}$ be a partition of $[a, b]$.
The p-variation of f is the quantity

$$V_p f = \left(\sup_\tau \sum_{k=1}^{n} |f(t_k) - f(t_{k-1})|^p \right)^{\frac{1}{p}}.$$

A function of bounded p-variation is a function whose p-variation is bounded. Let BV_p be the class that contains all these functions. For $p = 1$, $BV_1 = BV$ is the well-known class of functions of bounded variation.

Theorem

$$F_1 = BV \bigcap C[a,b]$$

Theorem

$$\forall p \geqslant 1 \quad BV_p \bigcap C[a,b] \subseteq F_{2-\frac{1}{p}}$$

Theorem

$$\forall 1 < \alpha \leqslant 2 \quad \forall q \geqslant 1 \quad \exists f \in F_\alpha \quad f \notin BV_q$$

Theorem

$$\forall p > 1 \quad \forall \delta > 0 \quad \exists T \in BV_{p+\delta} \bigcap C[a,b] \quad T \notin F_{2-\frac{1}{p}}$$

Theorem

(Salem)

$\forall p \geqslant 1 \ \ \forall f \in BV_p \bigcap C[0,2\pi] \ \ \forall x \in C[0,2\pi] \ \ S_n(f,x) \to f(x)$

Theorem

$\forall x_0 \in [0,2\pi] \ \ \forall \mu : F_1 \neq F^\mu \ \ \exists f \in F^\mu \ \ S_{n_k}(f,x_0) \to +\infty$