Some extremal problems for Fourier transform on hyperboloid

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Extremal problems for Fourier transform on $\mathbb{R}^d$

- Let $\mathcal{F}(f)(y) = \int_{\mathbb{R}^d} f(x)e^{-i(x,y)}\,dx$ be the Fourier transform.
- **Turan problem.** For central symmetric convex body $V \subset \mathbb{R}^d$

it is necessary to calculate the quantity

$$T(V, \mathbb{R}^d) = \sup \int_{\mathbb{R}^d} f(x)\,dx,$$

$$\text{if} \quad f \in C_b(\mathbb{R}^d), \quad f(0) = 1, \quad \operatorname{supp} f \subset V, \quad \mathcal{F}(f)(y) \geqslant 0.$$

- Euclidean ball: C.L. Siegel (1935, $d \geqslant 1$, [1]),

R.P. Boas and M. Kac (1945, $d = 1$, [2]),
D.V. Gorbachev (2001, $d > 1$, [3]),
M.N. Kolountzakis and Sz.Gy. Révész (2003, $d > 1$, [6])

- Another bodies:

V.V. Arestov and E.E. Berdysheva (2001, 2002, tiles polytopes, [4, 5]), M.N. Kolountzakis and Sz.Gy. Révész (2003, spectral domains, [6, 7, 8])

- In all known cases:

$$T(V, \mathbb{R}^d) = \left|\frac{1}{2}V\right| = \int_{\frac{1}{2}V} dx, \quad f_V = \chi_{\frac{1}{2}V} * \chi_{\frac{1}{2}V}.$$

- **Fejér problem.** For central symmetric convex body $V \subset \mathbb{R}^d$ it is necessary to calculate the quantity

$$F(V, \mathbb{R}^d) = \sup g(0),$$

if

$$g \in L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d), \quad g(y) \geqslant 0,$$
$$\frac{1}{(2\pi)^d} \int_{\mathbb{R}^d} g(y)\, dy = 1, \quad \operatorname{supp} \mathcal{F}^{-1}(g) \subset V.$$

- **Remark.** By Paley-Wiener theorem the set of admissible functions coincides with the set of nonnegative entire functions of exponential type, defined by the dual body.

- $$T(V, \mathbb{R}^d) = F(V, \mathbb{R}^d).$$

- L. Fejér (1915, [9]), R.P. Boas and M. Kac (1945, $d = 1$, [2])

- **Delsarte problem.** Calculate the quantity

$$D(B_s, \mathbb{R}^d) = \sup \int_{\mathbb{R}^d} f(x)\, dx,$$

if $f \in L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d),\ f(0) = 1,\ f(x) \leqslant 0,\ |x| \geqslant s,\ \mathcal{F}(f)(y) \geqslant 0.$

- M. Viazovska (2016, d=8, [10]),

H. Cohn, A. Kumar, S.D. Miller, D. Radchenko, M. Viazovska (2016, d=24, [11])

- **Modified Delsarte problem.** Calculate the quantity

$$D(E_1^r, B_s, \mathbb{R}^d) = \sup \int_{\mathbb{R}^d} g(y)\, dy,$$

$$\text{if} \quad g \in L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d), \quad g(0) = 1, \quad g(y) \leqslant 0,\ |y| \geq s,$$
$$\operatorname{supp} \mathcal{F}^{-1}(g) \subset B_r, \quad \mathcal{F}^{-1}(g)(y) \geqslant 0.$$

- Unique case: $r = \frac{2q_{d/2}}{s}, \quad J_{d/2}(q_{d/2}) = 0.$

- V.I. Levenshtein (1979, [12]), V.A. Yudin (1989, [13]),

D.V. Gorbachev (2000, [14]), H. Cohn (2002, [15])

- **Bohman problem.** Calculate the quantity

$$B(B_r, \mathbb{R}^d) = \inf \int_{\mathbb{R}^d} |y|^2 g(y)\, dy,$$

if

$$g \in L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d),\ g(y) \geqslant 0,\ \int_{\mathbb{R}^d} g(y)\, dy = 1,\ \operatorname{supp} \mathcal{F}^{-1}(g) \subset B_r.$$

- H. Bohman (1960, $d = 1$, [16]), V. A. Yudin (1976, $d > 1$, [17]), W. Ehm, T. Gneiting, D. Richards (2004, $d > 1$, [18])
- Let g be real continuous function, and let $\Lambda(g) = \sup\{|y| : g(y) > 0\}$.
- **Logan problem.** Calculate the quantity

$$L(B_r, \mathbb{R}^d) = \inf \Lambda(g),$$

if

$$g \in L^1(\mathbb{R}^d) \cap C_b(\mathbb{R}^d), \quad g \not\equiv 0, \quad \operatorname{supp} \mathcal{F}^{-1}(g) \subset B_r, \quad \mathcal{F}^{-1}(g)(y) \geqslant 0, .$$

- B.F. Logan (1983, $d = 1$, [19, 20]), N.I. Chernykh (1967, $d = 1$, [21]), V.A. Yudin (1981, $d > 1$, [22]), D.V. Gorbachev (2000, $d > 1$, [23]), E.E. Berdysheva (1999, cube, [24])

Extremal problems for Hankel transform on $\mathbb{R}_+$

- Extremal functions in these extremal problems for the ball are radial. By averaging functions over the Euclidean sphere the problems are reduced to analogous problems for the Hankel transform.

- Let $\alpha \geqslant -1/2$, and suppose that $J_\alpha(t)$ is the Bessel function of the order α,

$$j_\alpha(t) = 2^\alpha \Gamma(\alpha+1) \frac{J_\alpha(t)}{t^\alpha} \quad \left(j_{d/2-1}(t) = \int_{\mathbb{S}^{d-1}} e^{i(x,\xi)}\, d\omega(\xi),\ |x| = t \right)$$

is the normalized Bessel function, q_α is minimal positive zero of J_α,

$$d\nu_\alpha(t) = (2^\alpha \Gamma(\alpha+1))^{-1} t^{2\alpha+1}\, dt$$

is the power measure on the half-line $\mathbb{R}_+$, and

$$\mathcal{H}_\alpha(\lambda) = \int_0^\infty f(t) j_\alpha(\lambda t) d\nu_\alpha(t)$$

is the Hankel transform. Note that $\mathcal{H}_\alpha^{-1} = \mathcal{H}_\alpha$. The restriction of the Fourier transform on radial functions leads to the Hankel transform with $\alpha = \frac{d}{2} - 1$.

- Let $\chi_r(t)$ be characteristic function of the segment $[0, r]$.
- **Turan problem.** Calculate the quantity

$$T_\alpha(r, \mathbb{R}_+) = \sup \int_0^\infty f(t)\, d\nu_\alpha(t),$$

$$\text{if} \quad f \in C_b(\mathbb{R}_+), \quad f(0) = 1, \quad \operatorname{supp} f \subset [0, r], \quad \mathcal{H}_\alpha(f)(\lambda) \geqslant 0.$$

- **Fejér problem.** Calculate the quantity

$$F_\alpha(r, \mathbb{R}_+) = \sup g(0),$$

$$\text{if} \quad g \in L^1(\mathbb{R}_+, d\nu_\alpha) \cap C_b(\mathbb{R}_+), \quad g(y) \geqslant 0,$$

$$\int_0^\infty g(\lambda)\, d\nu_\alpha(\lambda) = 1, \quad \operatorname{supp} \mathcal{H}_\alpha(g) \subset [0, r].$$

- **Remark.** By Paley-Wiener theorem for the Hankel transform the set of admissible functions coincides with the set of even nonnegative entire functions of exponential type at most r.
- **Theorem 1.** $T_\alpha(r, \mathbb{R}_+) = F_\alpha(r, \mathbb{R}_+) = \int_0^{r/2} d\nu_\alpha(t)$ and

$$f_r(t) = (\chi_{r/2} * \chi_{r/2})(t), \quad g_r(\lambda) = c\mathcal{H}_\alpha(f_r)(\lambda) = j_{\alpha+1}^2(\lambda r/2).$$

- **Delsarte problem.** Calculate the quantity

$$D_\alpha(s,\mathbb{R}_+) = \sup \int_0^\infty f(t)\, d\nu_\alpha(t),$$

if

$$f \in L^1(\mathbb{R}_+, d\nu_\alpha) \cap C_b(\mathbb{R}_+),\ f(0)=1,\ f(t) \le 0,\ t \geqslant s,\ \mathcal{H}_\alpha(f)(\lambda) \geqslant 0.$$

- This problem is solved only for $\alpha = -1/2$, 3, 11.
- **Modified Delsarte problem.** Calculate the quantity

$$D_\alpha(r,s,\mathbb{R}_+) = \sup \int_0^\infty g(\lambda)\, d\nu_\alpha(\lambda),$$

if

$$g \in L^1(\mathbb{R}_+, d\nu_\alpha) \cap C_b(\mathbb{R}_+), \quad g(0)=1, \quad g(\lambda) \leqslant 0,\ \lambda \geqslant s,$$
$$\operatorname{supp} \mathcal{H}_\alpha(g) \subset [0,r], \quad \mathcal{H}_\alpha(g)(\lambda) \geqslant 0.$$

- **Theorem 2.** $D_\alpha(r, \frac{2q_{\alpha+1}}{r}, \mathbb{R}_+) = \left(\int_0^{r/2} d\nu_\alpha(\lambda)\right)^{-1}$ and

$$g_r(\lambda) = \frac{j_{\alpha+1}^2(\lambda r/2)}{1-\left(\lambda r/2q_{\alpha+1}\right)^2}.$$

- **Bohman problem.** Calculate the quantity

$$B_\alpha(r,\mathbb{R}_+) = \inf \int_0^\infty \lambda^2 g(\lambda)\, d\nu_\alpha(\lambda),$$

if

$$g \in L^1(\mathbb{R}_+, d\nu_\alpha) \cap C_b(\mathbb{R}_+), \quad g(\lambda) \geqslant 0,$$
$$\int_0^\infty g(\lambda)\, d\nu_\alpha(\lambda) = 1, \quad \operatorname{supp} \mathcal{H}_\alpha(g) \subset [0,r].$$

- **Theorem 3.** $B_\alpha(r,\mathbb{R}_+) = \left(\frac{2q_\alpha}{r}\right)^2$ and

$$g_r(\lambda) = \frac{j_\alpha^2(\lambda r/2)}{\left(1 - \left(\lambda r/2q_\alpha\right)^2\right)^2}.$$

- Let g be real continuous function, and let $\Lambda(g) = \sup\{\lambda : \, g(\lambda) > 0\}$.

- **Logan problem.** Calculate the quantity

$$L_\alpha(r, \mathbb{R}_+) = \inf \Lambda(g),$$

if

$$g \in L^1(\mathbb{R}_+, d\nu_\alpha) \cap C_b(\mathbb{R}_+), \quad g(\lambda) \not\equiv 0,$$
$$\operatorname{supp} \mathcal{H}_\alpha(g) \subset [0, r], \quad \mathcal{H}_\alpha(g)(\lambda) \geqslant 0.$$

- **Theorem 4.** $L_\alpha(r, \mathbb{R}_+) = \frac{2q_\alpha}{r}$ and

$$g_r(\lambda) = \frac{j_\alpha^2(\lambda r/2)}{1 - \Big(\lambda r/2q_\alpha\Big)^2}.$$

- Theorems 1-4 were proved by D.V. Gorbachev ([14, 3, 23, 25, 26]). He proved the uniqueness of extremal functions.

- A unified method for solving of these problems is to use the Gauss and Markov quadrature formulae on the half-line with nodes at zeros of the Bessel function (C. Frappier and P. Oliver (1993, [27]), G.R. Grozev and Q.I. Rahman (1995, [28]), R.B. Ghanem and C. Frappier (1998, [29])).
- Let E_1^r be the set of even entire functions of exponential type at most r, whose restrictions on $\mathbb{R}_+$ belong to $L^1(\mathbb{R}_+, d\nu_\alpha)$, and let $0 < q_{\alpha,1} < \ldots < q_{\alpha,n} < \ldots$ be positive zeros of $J_\alpha(t)$.
- **Theorem 5.** *For any function $g \in E_1^r$ the Gauss quadrature formula with positive weights holds*

$$\int_0^\infty g(\lambda)\,d\nu_\alpha(\lambda) = \sum_{k=1}^\infty \gamma_{\alpha,k}(r) g(2q_{\alpha,k}/r). \tag{1}$$

The series in (1) converges absolutely.

- **Theorem 6.** *For any function $g \in E_1^r$ the Markov quadrature formula with positive weights holds*

$$\int_0^\infty g(\lambda)\,d\nu_\alpha(\lambda) = \gamma'_{\alpha,0}(r) g(0) + \sum_{k=1}^\infty \gamma'_{\alpha,k}(r) g(2q_{\alpha+1,k}/r). \tag{2}$$

The series in (2) converges absolutely.

- Let us give an example of the application of the Gauss quadrature formula in the solution of the Bohman problem. Since an admissible function $g \in E_1^r$, $\lambda^2 g \in E_1^r$, $g(\lambda) \geqslant 0$, and $\int_0^\infty g(\lambda)\, d\nu_\alpha(\lambda) = 1$, then applying the Gauss quadrature formula two times, we obtain

$$\begin{aligned}\int_0^\infty \lambda^2 g(\lambda)\, d\nu_\alpha(\lambda) &= \sum_{k=1}^\infty \gamma_{\alpha,k}(r)(2q_{\alpha,k}/r)^2 g(2q_{\alpha,k}/\tau)\\ &\geqslant (2q_{\alpha,1}/r)^2 \sum_{k=1}^\infty \gamma_{\alpha,k}(r) g(2q_{\alpha,k}/r)\\ &= (2q_{\alpha,1}/r)^2 \int_0^\infty g(\lambda)\, d\nu_\alpha(\lambda) = (2q_{\alpha,1}/r)^2.\end{aligned}$$

- The extremal function $g_r(\lambda)$ has at the points $2q_{\alpha,k}/r$, $k \geqslant 2$, doubling zeros, therefore the following function is extremizer

$$g_\tau(\lambda) = \frac{j_\alpha^2(\lambda r/2)}{\Big(1 - \Big(\lambda r/2q_\alpha\Big)^2\Big)^2}.$$

- Recently (2015, [30]) we proved the Gauss and Markov quadrature formulae on the half-line with nodes at zeros of eigenfunctions of the Shturm–Lioville problem under some natural conditions on weight function w, which, in particular, are fulfilled for the power weight $w(t) = t^{2\alpha+1}$, $\alpha \geqslant -1/2$, and hyperbolic weight

$$w(t) = (\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}, \quad \alpha \geqslant \beta \geqslant -1/2.$$

- Let $\lambda_0 \geqslant 0$, and suppose that the Shturm–Lioville problem

$$\frac{\partial}{\partial t}\Big(w(t)\,\frac{\partial}{\partial t}\,u_\lambda(t)\Big) + \big(\lambda^2 + \lambda_0^2\big)w(t)u_\lambda(t) = 0,$$
$$u_\lambda(0) = 1, \quad \frac{\partial u_\lambda}{\partial t}\,(0) = 0, \quad \lambda, t \in \mathbb{R}_+,$$

has spectral measure $d\sigma(\lambda) = s(\lambda)\,d\lambda$, $s(\lambda) \asymp \lambda^{2\alpha+1}$, $\lambda \to +\infty$, and an eigenfunction $\varphi(t,\lambda)$, which is an even and analytic function of t on $\mathbb{R}$ and even entire function of exponential type $|t|$ with respect to λ. Let $0 < \lambda_1(t) < \ldots < \lambda_k(t) < \ldots$ be positive zeros of $\varphi(t,\lambda)$ with respect to λ.

- Let $\varphi_0(t) = \varphi(t, 0)$, let $u(t, \lambda) = \varphi(t, \lambda)/\varphi_0(t)$, let $0 < \lambda'_1(t) < \ldots < \lambda'_k(t) < \ldots$ be positive zeros of $\frac{\partial}{\partial t} u(t, \lambda)$ with respect to λ, and let E_1^r be the set of even entire functions of exponential type at most r, whose restrictions on $\mathbb{R}_+$ belong to $L^1(\mathbb{R}_+, d\sigma)$.

- **Theorem 7.** *For any function $g \in E_1^r$ the Gauss quadrature formula with positive weights holds*

$$\int_0^\infty g(\lambda)\, d\sigma(\lambda) = \sum_{k=1}^\infty \gamma_k(r) g(\lambda_k(r/2)). \tag{3}$$

The series in (3) converges absolutely.

- **Theorem 8.** *For any function $g \in E_1^r$ the Markov quadrature formula with positive weights holds*

$$\int_0^\infty g(\lambda)\, d\sigma(\lambda) = \gamma'_0(r) g(0) + \sum_{k=1}^\infty \gamma'_k(r) g(\lambda'_k(r/2)). \tag{4}$$

The series in (4) converges absolutely.

Extremal problems for Jacobi transform on $\mathbb{R}_+$

- In the case of hyperbolic weight

$$w(t) = 2^{2\rho}(\sinh t)^{2\alpha+1}(\cosh t)^{2\beta+1}, \quad t \in \mathbb{R}_+, \quad \alpha \geq \beta \geq -1/2,$$

where $\rho = \alpha + \beta + 1 = \lambda_0$, eigenfunction $\varphi_\lambda(t)$ is the Jacobi function

$$\varphi_\lambda(t) = F\Big(\frac{\rho + i\lambda}{2}, \frac{\rho - i\lambda}{2}; \alpha + 1; -(\sinh t)^2\Big).$$

- Let $d\mu(t) = w(t)\,dt$, and let $d\sigma(\lambda) = s(\lambda)\,d\lambda$,

$$s(\lambda) = (2\pi)^{-1}\left|\frac{2^{\rho - i\lambda}\Gamma(\alpha+1)\Gamma(i\lambda)}{\Gamma((\rho+i\lambda)/2)\Gamma((\rho+i\lambda)/2-\beta)}\right|^{-2},$$

be the spectral measure. The direct and inverse Jacobi transforms are defined by equalities

$$\mathcal{J}f(\lambda) = \int_0^\infty f(t)\varphi_\lambda(t)\,d\mu(t), \quad \mathcal{J}^{-1}g(t) = \int_0^\infty g(\lambda)\varphi_\lambda(t)\,d\sigma(\lambda).$$

- **Turan problem.** Calculate the quantity

$$T_{\alpha,\beta}(r,\mathbb{R}_+)=\sup\mathcal{J}(f)(0)=\sup\int_0^\infty f(t)\varphi_0(t)\,d\mu(t),$$

$$\text{if}\quad f\in C_b(\mathbb{R}_+),\quad f(0)=1,\quad \operatorname{supp} f\subset[0,r],\quad \mathcal{J}(f)(\lambda)\geqslant 0.$$

- **Fejér problem.** Calculate the quantity

$$F_{\alpha,\beta}(r,\mathbb{R}_+)=\sup g(0),$$

$$\text{if}\quad g\in L^1(\mathbb{R}_+,d\sigma)\cap C_b(\mathbb{R}_+),\quad g(\lambda)\geqslant 0,$$

$$\int_0^\infty g(\lambda)\,d\sigma(\lambda)=1,\quad \operatorname{supp}\mathcal{J}^{-1}(g)\subset[0,r].$$

- **Remark.** By Paley-Wiener theorem for the Jacobi transform the set of admissible functions coincides with the set of even nonnegative entire functions of exponential type at most r.
- Let $u_\lambda(t)=\varphi_\lambda(t)/\varphi_0(t)$, and let $\Delta(t)=\varphi_0^2(t)w(t)$.
- **Theorem 9.** [33] $T_{\alpha,\beta}(r,\mathbb{R}_+)=F_{\alpha,\beta}(r,\mathbb{R}_+)=\int_0^{r/2}\Delta(t)\,dt$ and

$$f_r(t)=(\varphi_0\chi_{r/2}*\varphi_0\chi_{r/2})(t),\quad g_r(\lambda)=c\mathcal{J}(f_r)(\lambda)=\left(\frac{\frac{\partial}{\partial t}u_\lambda(r/2)}{\lambda^2}\right)^2.$$

- **Delsarte problem.** Calculate the quantity

$$D_{\alpha,\beta}(s,\mathbb{R}_+) = \sup \mathcal{J}(f)(0) = \sup \int_0^\infty f(t)\varphi_0(t)\,d\mu(t),$$

if

$$f \in L_1(\mathbb{R}_+, d\mu) \cap C_b(\mathbb{R}_+),\ \ f(0)=1,\ \ f(t)\leqslant 0,\ \ t\geqslant s,\ \ \mathcal{J}(f)(\lambda)\geqslant 0.$$

- **Modified Delsarte problem for entire functions.** Calculate the quantity

$$D_{\alpha,\beta}(r,s,\mathbb{R}_+) = \sup \mathcal{J}^{-1}(g)(0) = \sup \int_0^\infty g(\lambda)\,d\sigma(\lambda),$$

$$\text{if}\quad g \in L^1(\mathbb{R}_+, d\sigma)\cap C_b(\mathbb{R}_+),\quad g(0)=1,\quad g(\lambda)\leqslant 0,\ \lambda\geqslant s,$$
$$\operatorname{supp}\mathcal{J}^{-1}(g)\subset[0,r],\quad \mathcal{J}^{-1}(g)(\lambda)\geqslant 0.$$

- **Theorem 10.** [31] $D_{\alpha,\beta}(r,\lambda_1'(r/2),\mathbb{R}_+) = \left(\int_0^{r/2}\Delta(t)\,dt\right)^{-1}$

and

$$g_\tau(\lambda) = \frac{\left(\lambda^{-2}\frac{\partial}{\partial t}u_\lambda(r/2)\right)^2}{1-\left(\lambda/\lambda_1'(r/2)\right)^2}.$$

- **Bohman problem.** Calculate the quantity

$$B_{\alpha,\beta}(r,\mathbb{R}_+) = \inf \int_0^\infty (\lambda^2+\rho^2) g(\lambda)\, d\sigma(\lambda),$$

if

$$g \in L^1(\mathbb{R}_+, d\sigma) \cap C_b(\mathbb{R}_+), \quad g(\lambda) \geqslant 0,$$
$$\int_0^\infty g(\lambda)\, d\sigma(\lambda) = 1, \quad \operatorname{supp} \mathcal{J}^{-1}(g) \subset [0, r].$$

- **Theorem 11.** [32] $B_{\alpha,\beta}(r,\mathbb{R}_+) = \lambda_1^2(\tau/2) + \rho^2$

and

$$g_r(\lambda) = \frac{\varphi_\lambda^2(r/2)}{\left(1 - \left(\lambda/\lambda_1(r/2)\right)^2\right)^2}.$$

- Recall that $\Lambda(g) = \sup\{\lambda : \, g(\lambda) > 0\}$.
- **Logan problem.** Calculate the quantity

$$L_{\alpha,\beta}(r, \mathbb{R}_+) = \inf \Lambda(g),$$

if

$$g \in L^1(\mathbb{R}_+, d\sigma) \cap C_b(\mathbb{R}_+), \quad g(\lambda) \not\equiv 0,$$
$$\operatorname{supp} \mathcal{J}^{-1}(g) \subset [0, r], \quad \mathcal{J}^{-1}(g)(\lambda) \geqslant 0.$$

- **Theorem 12.** [34] $\quad L_{\alpha,\beta}(r, \mathbb{R}_+) = \lambda_1(r/2)$

and

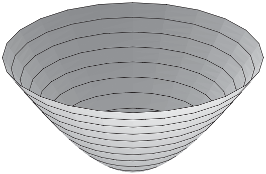
$$g_r(\lambda) = \frac{\varphi_\lambda^2(r/2)}{1 - \Big(\lambda/\lambda_1(r/2)\Big)^2}.$$

Extremal problems for Fourier transform on H^d

- Let $d \in \mathbb{N}$, $d \geq 2$, and suppose that $\mathbb{R}^d$ is d-dimensional real Euclidean space with inner product $(x, y) = x_1 y_1 + \ldots + x_d y_d$, and norm $|x| = \sqrt{(x, x)}$,

$$\mathbb{S}^{d-1} = \{x \in \mathbb{R}^d : |x| = 1\}$$

is the Euclidean sphere, $\mathbb{R}^{d,1}$ is $(d+1)$-dimensional real pseudoeuclidean space with bilinear form $[x, y] = -x_1 y_1 - \ldots - x_d y_d + x_{d+1} y_{d+1}$,



$$\mathbb{H}^d = \{x \in \mathbb{R}^{d,1} : [x, x] = 1,\ x_{d+1} > 0\}$$

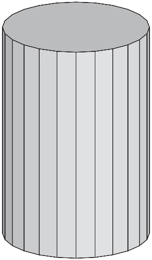
is the upper sheet of two sheets hyperboloid,

- $d(x, y) = \operatorname{arc\,cosh}[x, y] = \ln([x, y] + \sqrt{[x, y]^2 - 1})$ is the distance between $x, y \in \mathbb{H}^d$.
- The pair $\left(\mathbb{H}^d, d(\cdot, \cdot)\right)$ is known as the Lobachevskii space. Let $x_0 = (0, \ldots, 0, 1) \in \mathbb{H}^d$, $d(x, x_0) = d(x)$, $r > 0$, and let $B_\tau = \{x \in \mathbb{H}^{d-1} : d(x) \leq r\}$ be the ball in the Lobachevskii space.

- Let $t > 0$, $\eta \in \mathbb{S}^{d-1}$, $x = (\sinh t\, \eta, \cosh t) \in \mathbb{H}^d$, and let

$$d\mu(t) = w(t)\, dt = 2^{d-1} \sinh^{d-1} t\, dt, \quad d\omega(\eta) = \frac{1}{|\mathbb{S}^{d-1}|}\, d\eta,$$

$$d\nu(x) = d\mu(t) d\omega(\eta)$$



be the Lebesgue measures on $\mathbb{R}_+$, $\mathbb{S}^{d-1}$ and $\mathbb{H}^d$, respectively. Note that $d\omega$ is the probability measure on the sphere, invariant under rotation group $SO(d)$ and the measure $d\nu$ is invariant under hyperbolic rotation group $SO_0(d, 1)$.

- Let $\lambda \in \mathbb{R}_+ = [0, \infty)$, $\xi \in \mathbb{S}^{d-1}$, $y = (\lambda, \xi) \in \mathbb{R}_+ \times \mathbb{S}^{d-1} = \Omega^d$, and let

$$d\sigma(\lambda) = s(\lambda)\, d\lambda = 2^{3-2d} \Gamma^{-2}\left(\frac{d}{2}\right) \left| \frac{\Gamma\left(\frac{d-1}{2} + i\lambda\right)}{\Gamma(i\lambda)} \right|^2 d\lambda,$$

$$d\tau(y) = d\sigma(\lambda) d\omega(\xi)$$

be the Lebesgue measures on $\mathbb{R}_+$ and Ω^d.

• The direct and inverse Fourier transforms are defined by equalities

$$\mathcal{F}f(y) = \int_{\mathbb{H}^d} f(x)[x,\xi']^{-\frac{d-1}{2}-i\lambda}\, d\nu(x),$$

$$\mathcal{F}^{-1}g(x) = \int_{\Omega^d} g(y)[x,\xi']^{-\frac{d-1}{2}+i\lambda}\, d\tau(y),$$

where $\xi' = (\xi, 1)$, $\xi \in \mathbb{S}^{d-1}$.

• Let

$$\varphi_\lambda(t) = F\Big(\frac{(d-1)/2 + i\lambda}{2}, \frac{(d-1)/2 - i\lambda}{2}; \frac{d}{2}; -(\sinh t)^2\Big)$$

be the Jacobi function $(\alpha = (d-2)/2,\ \beta = -1/2)$. We have

$$\varphi_\lambda(t) = \int_{\mathbb{S}^{d-1}} [x,\xi']^{-\frac{d-1}{2}\pm i\lambda}\, d\omega(\xi),$$

where $x = (\sinh t\, \eta, \cosh t)$, $\eta, \in \mathbb{S}^{d-1}$, $\xi' = (\xi, 1)$.

- Two averaging operators over sphere

$$Pf(t) = \int_{\mathbb{S}^{d-1}} f(x)\, d\omega(\eta),\ \ x = (\sinh t\, \eta, \cosh t) \in \mathbb{H}^d,$$

$$Qg(\lambda) = \int_{\mathbb{S}^{d-1}} g(y)\, d\omega(\xi),\ \ y = (\lambda, \xi) \in \Omega^d$$

give us spherical functions on $\mathbb{H}^d$ and Ω^d. They are used both for the setting and for the solving of extremal problems.

- If $f(x) = f_0(d(x)) = f_0(t)$ and $g(y) = g_0(\lambda)$ are spherical functions, then

$$\mathcal{F}f(y) = \mathcal{J}f_0(\lambda), \quad \mathcal{F}^{-1}g(x) = \mathcal{J}^{-1}g_0(t).$$

- Let $\Delta(t) = \varphi_0^2(t)w(t)$, $u_\lambda(t) = \varphi_\lambda(t)/\varphi_0(t)$.

- Some facts from the harmonic analysis on the hyperboloid can be found in [35].

- **Turán problem**. Calculate the quantity

$$T(r, \mathbb{H}^d) = \sup Q(\mathcal{F}f)(0),$$

if

$$f \in C_b(\mathbb{H}^d), \quad f(x_0) = 1, \quad \operatorname{supp} f \subset B_r, \quad \mathcal{F}f(y) \geqslant 0.$$

- **Fejér problem**. Calculate the quantity

$$F(r, \mathbb{H}^d) = \sup Qg(0),$$

if

$$g \in L^1(\Omega^d, d\tau) \cap C_b(\Omega^d), \quad g(y) \geqslant 0,$$
$$\int_{\Omega^d} g(y)\, d\tau(y) = 1, \quad \operatorname{supp} \mathcal{F}^{-1}(g) \subset B_r.$$

- **Remark.** Admissible functions in the Fejér problem are even entire functions of exponential type at most r with respect to λ.
- **Theorem 13.** [34] $T(r, \mathbb{H}^d) = F(r, \mathbb{H}^d) = \int_0^{r/2} \Delta(t)\, dt$ and

$$f_r(x) = (\varphi_0\chi_{r/2} * \varphi_0\chi_{r/2})(t), \quad g_r(y) = c\mathcal{F}(f_r)(y) = \left(\frac{\frac{\partial}{\partial t} u_\lambda(r/2)}{\lambda^2}\right)^2,$$

$$x = (\sinh t\, \eta, \cosh t) \in \mathbb{H}^d, \quad y = (\lambda, \xi) \in \Omega^d.$$

- **Delsarte problem.** Calculate the quantity

$$D(s,\mathbb{H}^d) = \sup Q(\mathcal{F}f)(0),$$

if

$$f \in \mathbb{H}^d,\ f(x_0)=1,\ f(x) \leqslant 0,\ d(x) \geqslant s,\ \mathcal{F}(f)(y) \geqslant 0.$$

- **Modified Delsarte problem.** Calculate the quantity

$$D(r,s,\mathbb{H}^d) = \sup \int_{\Omega^d} g(y)\,d\tau(y),$$

if

$$g \in L^1(\Omega^d, d\tau) \cap C_b(\Omega^d), \quad Qg(0)=1, \quad g(\lambda,\xi) \leqslant 0,\ \ \lambda \geqslant s,$$
$$\operatorname{supp} \mathcal{F}^{-1}(g) \subset B_r, \quad \mathcal{F}^{-1}(g)(x) \geqslant 0.$$

- **Theorem 14.** [34] $D(r,\lambda_1'(r/2),\mathbb{H}^d) = \left(\int_0^{r/2} \Delta(t)\,dt\right)^{-1}$

and

$$g_r(y) = \frac{\left(\lambda^{-2}\frac{\partial}{\partial t}u_\lambda(r/2)\right)^2}{1-\left(\lambda/\lambda_1'(r/2)\right)^2}, \quad y=(\lambda,\xi)\in\Omega^d.$$

- Let $\rho = \alpha + \beta + 1 = (d-2)/2 - 1/2 + 1 = (d-1)/2$.
- **Bohman problem.** Calculate the quantity

$$B(r, \mathbb{H}^d) = \inf \int_{\Omega^d} (\lambda^2 + \rho^2) g(y)\, d\tau(y), \quad y = (\lambda, \xi),$$

if

$$g \in L^1(\Omega^d, d\tau) \cap C_b(\Omega^d), \quad g(y) \geqslant 0,$$

$$\int_{\Omega^d} g(y)\, d\tau(y) = 1, \quad \operatorname{supp} \mathcal{F}^{-1}(g) \subset B_r.$$

- **Theorem 15.** [34] $B(r, \mathbb{H}^d) = \lambda_1^2(r/2) + \rho^2$

and

$$g_r(y) = \frac{\varphi_\lambda^2(r/2)}{\left(1 - \left(\lambda/\lambda_1(r/2)\right)^2\right)^2}, \quad y = (\lambda, \xi) \in \Omega^d.$$

- Let $y = (\lambda, \xi) \in \Omega^d$, let $g(y)$ be a real, continuous function on Ω^d, and let

$$\Lambda(g) = \sup\{\lambda > 0 : \ g(\lambda, \xi) > 0, \ \xi \in \mathbb{S}^{d-1}\}.$$

- **Logan problem.** Calculate the quantity

$$L(r, \mathbb{H}^d) = \inf \Lambda(g),$$

if

$$g \in L^1(\Omega^d, d\tau) \cap C_b(\Omega^d), \quad g(y) \not\equiv 0,$$

$$\operatorname{supp} \mathcal{F}^{-1}(g) \subset B_r \quad \mathcal{F}^{-1}(g)(x) \geqslant 0.$$

- **Theorem 16.** [34] $L(r, \mathbb{H}^d) = \lambda_1(r/2)$

and

$$g_r(y) = \frac{\varphi_\lambda^2(r/2)}{1 - \Big(\lambda/\lambda_1(r/2)\Big)^2}, \quad y = (\lambda, \xi) \in \Omega^d.$$

Thank you for attention to the talk!

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