# Szegö–Taikov inequality for conjugate polynomials

Polina Glazyrina

Ural Federal University
Yekaterinburg, Russia

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# Problem

$$\mathscr{F}_n: \quad f_n(t) = a_0 + \sum_{k=1}^{n} (a_k \cos kt + b_k \sin kt), \quad a_k, b_k \in \mathbb{C}$$

the conjugate to $f_n$: $\widetilde{f}_n(t) = \sum_{k=1}^{n} (a_k \sin kt - b_k \cos kt)$

We study the operator

$$\Lambda_{\theta,A} f_n(t) = Aa_0 + \sum_{k=1}^{n} (a_k \cos(kt+\theta) + b_k \sin(kt+\theta))$$

$$= Aa_0 + \cos\theta (f_n(t) - a_0) - \sin\theta \widetilde{f}_n(t), \quad \theta \in \mathbb{R},\ A \in \mathbb{C}$$

$$\left\| \Lambda_{\theta,A} f_n \right\|_\infty \leqslant C_n(\theta, A) \|f_n\|_\infty, \quad f_n \in \mathscr{F}_n.$$

# Some values of $\Lambda_{\theta,A}$

$$\begin{aligned}\Lambda_{\theta,A} f_n(t) &= A a_0 + \sum_{k=1}^{n} (a_k \cos(kt+\theta) + b_k \sin(kt+\theta)) \\ &= A a_0 + \cos\theta (f_n(t) - a_0) - \sin\theta\, \widetilde{f}_n(t)\end{aligned}$$

$$\Lambda_{0,1} f_n = f_n, \qquad \Lambda_{0,0} f_n = f_n - a_0, \qquad \Lambda_{3\pi/2,0} f_n = \widetilde{f}_n$$

$$\Lambda_{\theta,\cos\theta} f_n(t) = D^0 \left( \cos\theta\, f_n(t) - \sin\theta\, \widetilde{f}_n(t) \right)$$

Weyl derivative

$$D^\alpha f_n(t) = \sum_{k=1}^{n} k^\alpha \left( a_k \cos(kt + \alpha\pi/2) + b_k \sin(kt + \alpha\pi/2) \right).$$

# Known estimates for $C_n(\theta, A)$

$$\Lambda_{\theta,A} f_n(t) = Aa_0 + \cos\theta(f_n(t) - a_0) - \sin\theta \widetilde{f}_n(t)$$

$$\left\|\Lambda_{\theta,A} f_n\right\|_\infty \leqslant C_n(\theta, A)\|f_n\|_\infty, \quad f_n \in \mathscr{F}_n.$$

$$C_n(\theta, A) = \frac{2}{\pi}|\sin\theta| \ln n + O(1) \text{ as } n \to \infty.$$

$C_{2m-1}(3\pi/2, 0)$ is Lebesgue constant for Lagrange interpolation polynomial based on the zeros of the Chebyshev polynomial of the first kind of degree $m$.

At present, two values of $C_n(\theta, A)$ are known:

Fejer, 1913: $\|f_n - a_0\|_\infty \leqslant C_n(0, 0)\|f_n\|_\infty,$

Szegö, 1943: $\|\widetilde{f}_n\|_\infty \leqslant C_n(3\pi/2, 0)\|f_n\|_\infty.$

# The result of Fejer for $C(0,0)$

In 1913, Fejér proved that for any $f_n \geqslant 0$, $a_0 > 0$

$$f_n \leqslant a_0(n+1).$$

He also established the relation

$$M \leqslant nm,$$

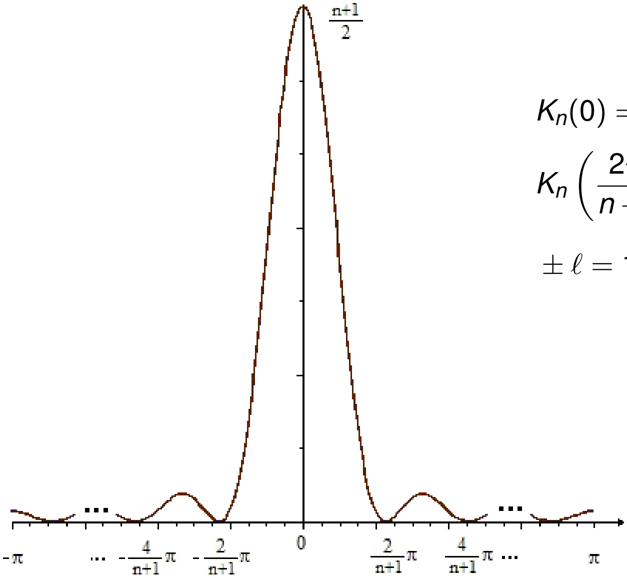
$M$ is the maximum, $-m$ is the minimum of a real polynomial $f_n$ with $a_0 = 0$. These results imply that

$$\|f_n - a_0\|_\infty \leqslant \frac{2n}{n+1}\|f_n\|_\infty,$$

the extremal polynomial is $K_n(t) - \|K_n\|_\infty/2$,
$K_n$ is the Fejér kernel

$$K_n(t) = \frac{1}{2} + \sum_{k=1}^{n}\left(1 - \frac{k}{n+1}\right)\cos kt = \frac{1}{2(n+1)}\left(\frac{\sin(n+1)t/2}{\sin t/2}\right)^2.$$

# The Fejér kernel




$$K_n(0) = \frac{n+1}{2}$$

$$K_n\left(\frac{2\pi\ell}{n+1}\right) = 0,$$

$$\pm\,\ell = 1, \ldots, \left[\frac{n+1}{2}\right]$$

# The result of Szegö for $C(3\pi/2, 0)$

In 1943, Szegö proved that

$$\|\widetilde{f_n}\|_\infty \leqslant \frac{2}{n+1} \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} \cot\left(\frac{\pi + 2\pi\ell}{2(n+1)}\right) \|f_n\|_\infty.$$

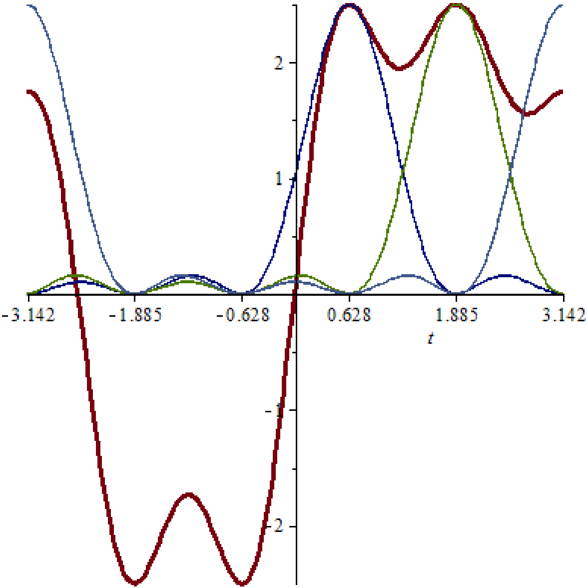
Szegö found all extremal polynomials with real coefficients. More precisely, for odd $n$, an extremal polynomial is unique (up to a non-zero constant factor and a shift of argument) and it is

$$f_n^*(t) = \sum_{\ell=0}^{\lfloor \frac{n-1}{2} \rfloor} K_n\left(t - \frac{\pi + 2\pi\ell}{n+1}\right) - K_n\left(t + \frac{\pi + 2\pi\ell}{n+1}\right).$$

For even $n$, extremal polynomials are given by the relation

$$f_n^{**}(t) = f_n^*(t) + \gamma K_n(t - \pi), \quad \gamma \in \mathbb{R}, \quad |\gamma| \leqslant 1.$$

# The extremal polynomial for $n = 4$, $\gamma = 0.7$



# Interpolation formula

For any $\theta \in \mathbb{R}$, $A \in \mathbb{C}$, and $f_n \in \mathscr{F}_n$

$$\Lambda_{\theta,A} f_n(t) = \frac{1}{n+1} \sum_{\ell=0}^{n} (q_\ell + A) \cdot f_n(t + t_\ell),$$

where $t_\ell = \dfrac{2\theta + 2\pi\ell}{n+1}$, $\ell = 0, \dots, n$,

$$q_\ell = -\frac{\sin(t_\ell/2 - \theta)}{\sin(t_\ell/2)} = -\cos\theta + \sin\theta \cot t_\ell/2, \qquad \theta \neq 0 \bmod \pi,$$

$$q_0 = n\cos\theta, \quad q_\ell = -\cos\theta, \quad \ell = 1, \dots, n, \qquad \theta = 0 \bmod \pi.$$

If $\theta \neq 0 \bmod \pi$ or $A \neq \cos\theta$, then at least $n$ of the coefficients $q_\ell + A$ do not vanish. In particular, $q_\ell = 0$ if and only if

$$1 \leqslant \ell \leqslant n-1 \quad \textit{and} \quad \theta = \frac{\pi\ell}{n} \text{ or}$$

$$0 \leqslant \ell \leqslant n-2 \quad \textit{and} \quad \ell = \frac{\theta n}{\pi} - n - 1.$$

# Theorem

Let $\theta \in [0, 2\pi)$ and $A \in \mathbb{C}$, then

$$\left\|\Lambda_{\theta,A} f_n\right\|_\infty \leqslant \frac{1}{n+1} \sum_{\ell=0}^{n} |q_\ell + A| \|f_n\|_\infty, \quad f_n \in \mathscr{F}_n. \tag{1}$$

If $\theta \neq 0 \bmod \pi$ or $A \neq \cos\theta$, then inequality (1) turns into an equality only for polynomials

$$f_n^*(t) = c \sum_{\ell=0}^{n} s_\ell K_n(t - t^* - t_\ell), \quad t^* \in \mathbb{R},\ c \in \mathbb{C},$$

where $s_\ell = \operatorname{sign}(q_\ell + A)$ if $q_\ell + A \neq 0$ and $s_\ell$ is an arbitrary complex number such that $|s_\ell| \leqslant 1$ if $q_\ell + A = 0$.

Thank you for your attention!

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