Convergence of subsequences of partial sums of trigonometric Fourier series

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The trigonometric system: $\left(\frac{1}{\sqrt{2\pi}}e^{\imath nx} \quad n = 0, \pm 1, \pm 2, \dots\right)$

$(x \in \mathbb{R}, \imath = \sqrt{-1})$. Orthonormal over any interval of length 2π. Let $T := [-\pi, \pi]$.

Let $f \in L^1(T)$. The kth Fourier coefficient of f:

$$\hat{f}(k) := \frac{1}{2\pi}\int_T f(x)e^{-\imath kt}dt,$$

$k \in \mathbb{Z}$.

The nth ($n \in \mathbb{N}$) partial sum of the Fourier series of f:

$$S_n f(y) := \sum_{k=-n}^{n} \hat{f}(k)e^{\imath ky}.$$

The trigonometric system

The nth ($n \in \mathbb{N}$) Fejér or $(C, 1)$ mean of function f:

$$\sigma_n f(y) := \frac{1}{n+1} \sum_{k=0}^{n} S_k f(y).$$

$$\sigma_n f(y) = \frac{1}{\pi} \int_T f(x) K_n(y-x) dx,$$

K_n is the nth Fejér kernel.

Lebesgue (1905, Mathematische Annalen):

For each integrable function a.e. convergence of Fejér means $\sigma_n f = \frac{1}{n+1} \sum_{k=0}^{n} S_k f \to f$.

Partial sums, first negative results, the trigonometric system

It is of main interest in the theory of trigonometric Fourier series that how to reconstruct the function from the partial sums of its Fourier series.

- Du Bois-Reymond (1876, Abhand. Akad. München) the Fourier series of a continuous function can unboundedly diverge at some point.
- Kolmogoroff (1923, Fund. Math.) constructed an example of a function $f \in L^1(T)$ such that the partial sums $S_m f(x)$ diverges unboundedly almost everywhere.
- Kolmogoroff (1926, C. R. Acad. Sci. Paris) there exists an integrable function with everywhere divergent Fourier series.

Partial sums, positive results, the trigonometric system

- Carleson (1966, Acta Math.) $f \in L^2(T)$, then $S_n f \to f$ almost everywhere.
- Hunt (1968, University Press, Carbondale, Ill.) $f \in L^p (p > 1)$ impl. a.e. conv.
- Antonov (1996, East J. Approx.) $f \in L \log^+ L \log^+ \log^+ \log^+ L$, $(\int |f| \log^+ |f| \log^+ \log^+ \log^+ |f| < \infty)$ then the partial sums converge to the function almost everywhere again.

What if we have only a subsequence of the partial sums? With respect to the partial sums and the Lebesgue space L^1 bad news...

- Totik (1982, Publicationes Mathematicae-Debrecen): for each subsequence (n_j) of the sequence of natural numbers there exists an integrable function f such that $\sup_j |S_{n_j} f| = +\infty$ everywhere. Moreover,
- Konyagin (2005, Proc. Steklov Inst. Math. Suppl.): for any increasing sequence (n_j) of positive integers and any nondecreasing function $\phi : [0, +\infty) \to [0, +\infty)$ satisfying condition $\phi(u) = o(u \log \log u)$, there is a function $f \in \phi(L)$ such that $\sup_j |S_{n_j} f| = +\infty$ everywhere.

Special subsequences?

Lacunary partial sums, the endpoint theorems, trigonometric system

- Di Plinio (2014, Collect. Math.)
  for (n_j) lacunary and $f \in L^1 \log^+ \log^+ L \log^+ \log^+ \log^+ \log^+ L$
  we have $S_{n_j} f \to f$ a.e.
- Victor Lie (2017, to appear in European Math. Journal)
  for any (n_j) lacunary and
  $\phi(u) = o(u \log^+ \log^+ u \log^+ \log^+ \log^+ \log^+ u)$, there exists a $f \in \phi(L)$ such that $\sup_j |S_{n_j} f| = +\infty$ everywhere.

What about the L^1 case? Some summation method is needed.

In 1936 Zalcwasser (1936, Stud. Math.) asked how fast can the sequence of integers (n_j) grow that it still holds:

$$\frac{1}{N}\sum_{j=1}^{N} S_{n_j} f \to f$$

a.e. for every function $f \in L^1$.

This problem with respect to the trigonometric system was completely solved for continuous functions and uniform convergence:

Salem, (1955, Am. J. Math.): If the sequence (n_j) is convex, then the condition $\sup_j j^{-1/2} \log n_j < +\infty$ is necessary and sufficient for the uniform convergence for every continuous function.

With respect to convergence almost everywhere, and integrable functions the situation is more complicated.

- In 1936 Zalcwasser (1936, Stud. Math.) proved the a.e. relation $\frac{1}{N}\sum_{j=1}^{N} S_{j^2}f \to f$ for each integrable function f.
- Salem, (1955, Am. J. Math.) writes that this theorem of Zalcwasser is extended to j^3 and j^4.
- Belinsky proved (1997, Proc. Am. Math. Soc.) the existence of a sequence $n_j \sim \exp(\sqrt[3]{j})$ such that the relation $\frac{1}{N}\sum_{j=1}^{N} S_{n_j}f \to f$ holds a.e. for every integrable function.
- Belinsky also conjectured that if the sequence (n_j) is convex, then the condition $\sup_j j^{-1/2}\log n_j < +\infty$ is necessary and sufficient again. So, that would be the answer for the problem of Zalcwasser (1936, Stud. Math.). (in this point of view (trigonometric system, a.e. convergence and L^1 functions.))

An example for kernel function, Fejér kernel

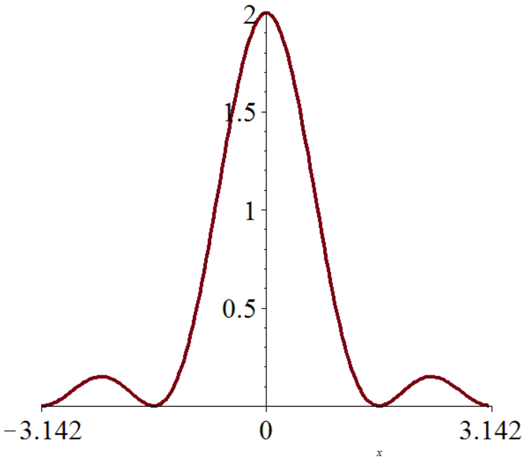

Figure: $\frac{1}{4}\sum_{k=0}^{3} D_k(x)$ Fejér kernel

An example for kernel function, Zalcwasser's kernel

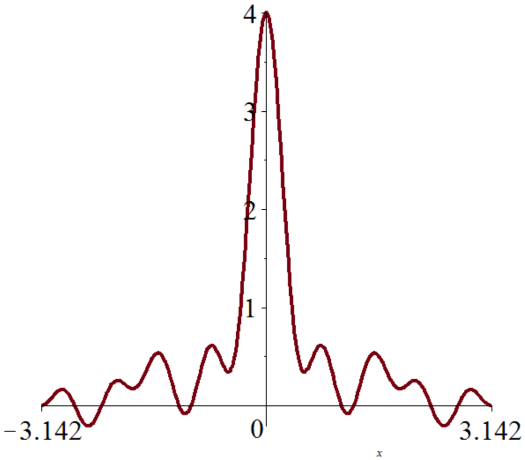

Figure: $\frac{1}{4}\sum_{k=0}^{3} D_{k^2}(x)$ Zalcwasser kernel

"Hopeless" to investigate a general Zalcwasser kernel.

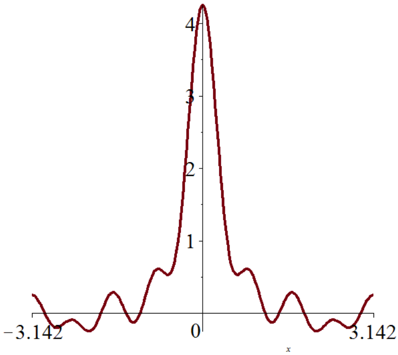

Figure: $\frac{1}{4}\sum_{k=0}^{3} D_{2^k}(x)$ kernel

$K_N \geq 0$ fails to hold and $\|K_N\|_1 \geq C\frac{\log n_N}{N}$ if $n_N \nearrow \infty$ fast enough.

Theorem (Gát, submitted)
Let $n_{j+1} \geq \left(1 + \frac{1}{j^\delta}\right) n_j$ for some $0 < \delta < \sqrt{5}/2 - 1/2 \approx 0.618$, $f \in L^1$. Then a.e.:

$$\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} S_{n_j} f = f.$$

Corollary Let (n_j) be a lacunary sequence of natural numbers. Then it holds the almost everywhere relation: $\lim_{N\to\infty} \frac{1}{N} \sum_{j=1}^{N} S_{n_j} f = f$ for every $f \in L^1(T)$.

Main tool: (Gát, submitted) Let $f \in L^1, \lambda > \|f\|_1$ and (n_j) lacunary: there exists: $F_1 \subset F_2 \subset \dots,$
mes $\bigcup F_j \leq C\|f\|_1/\lambda$ and

$$\left\| \frac{1}{N} \sum_{j=1}^{N} (S_{n_j} f - V_{n_j} f) \left(\sigma_{m_j} 1_{\overline{F_j}} \right) \right\|_2^2 \leq \frac{1}{N} C \log^5 N \|f\|_1 \lambda,$$

where $m_j \sim n_j$ ($V_n f$ is the nth de La Vallée Poussin mean).

Remark. Of course, without $\sigma_{m_j} 1_{\overline{F_j}}$ this inequlity does not hold for all $f \in L^1$. However, the mes. of F_j is "small", the mes. of $\overline{F_j}$ is "big" and $\sigma_{m_j} 1_{\overline{F_j}}$ is close to 1 on a "big" set.

The Walsh system

expansion with resp. the binary number system

$n = \sum_{i=0}^{\infty} n_i 2^i \in \mathbb{N}, \; x = \sum_{i=0}^{\infty} \frac{x_i}{2^{i+1}} \in [0,1) \quad n_i, x_i \in \{0,1\}$
$i = 0, 1, \ldots,$

If x is a dyadic rational number $(x \in \{\frac{p}{2^n} : p, n \in \mathbb{N}\})$ we choose the expansion which terminates in 0 's.

Walsh function

n-th Walsh-Paley function: $\omega_n(x) := (-1)^{\sum_{i=0}^{\infty} n_i x_i}$

Paley, *A remarkable series of orthogonal functions*, Proceedings of the London Mathematical Society (1932).

Can take $+1$ and -1 as a value.

Dirichlet and Fejér kernel functions:

Dirichlet and Fejér kernel functions

$$D_n := \sum_{k=0}^{n-1} \omega_k, \quad K_n := \frac{1}{n} \sum_{k=0}^{n-1} D_k,$$

Fourier coefficients, partial sums of Fourier series, Fejér means:

$$\hat{f}(n) := \int_0^1 f(x)\omega_n(x)dx \ (n \in \mathbb{N}),$$

$$S_n f(y) := \sum_{k=0}^{n-1} \hat{f}(k)\omega_k(y) = \int_0^1 f(x \dot{+} y) D_n(x) dx$$

$$\sigma_n f(y) := \frac{1}{n} \sum_{k=0}^{n-1} S_k f(y) = \int_0^1 f(x \dot{+} y) K_n(x) dx$$

Fejér means:

trigonometric system:

H. Lebesgue $\sigma_n f(x) \to f(x)$ for a.e. x. ,,Reconstruction the function"

Walsh case

Walsh-Paley system

N.J. Fine, Trans. Am. Math. Soc., 1949.

For the Walsh-Kaczmarz system

G. Gát. On (C; 1) summability of integrable functions with respect to the Walsh-Kaczmarz system. Stud. Math., 1998.

What does this Walsh-Kaczmarz system mean?

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Walsh-Kaczmarz system:

Let the Walsh-Kaczmarz functions be defined by $\kappa_0 = 1$ and for $n \geq 1, |n| = \lfloor \log_2 n \rfloor$

$$\kappa_n(x) := r_{|n|}(x)(-1)^{\sum_{k=0}^{|n|-1} n_k x_{|n|-1-k}}.$$

The Walsh-Paley system is $\omega := (\omega_n : n \in \mathbb{N})$ and the Walsh-Kaczmarz system is $\kappa := (\kappa_n : n \in \mathbb{N})$.

$$\{\kappa_n : 2^k \leq n < 2^{k+1}\} = \{\omega_n : 2^k \leq n < 2^{k+1}\}$$

for all $k \in \mathbb{N}$ and $\kappa_0 = \omega_0$. A dyadic blockwise ,,rearrangement".

Walsh-Paley Fejér kernel function

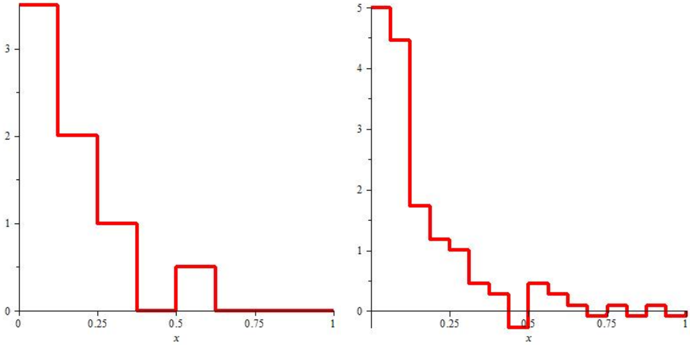

Figure: A K_8 and K_{11} Walsh-Paley-Fejér kernels

$K_n(x) \to 0$ $(n \to \infty)$ for every $x \neq 0$.

Walsh-Kaczmarz Fejér kernel function

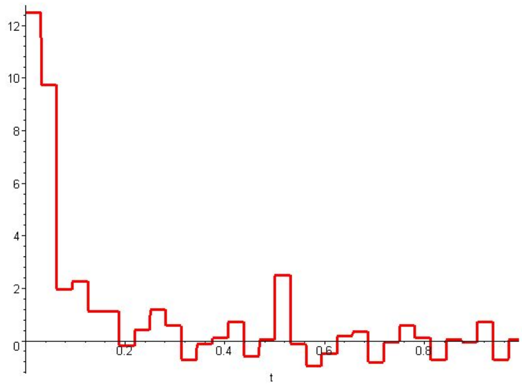

Figure: A K_{26} Walsh-Kaczmarz

$|K_n(x)| \to \infty$ $(n \to \infty)$ at every dyadic rational.

Theorem (Gát, JAT, 2010) Let (n_j) be a lacunary sequence of natural numbers. Then it holds the almost everywhere relation: $\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N} S_{n_j}f = f$ for every $f \in L^1(T)$.

Conjecture Let $n_{j+1} \geq \left(1+\frac{1}{j^\delta}\right) n_j$ for some $0<\delta<\sqrt{5}/2-1/2 \approx 0.618$, $f\in L^1$. Then a.e.:

$$\lim_{N\to\infty}\frac{1}{N}\sum_{j=1}^{N} S_{n_j}f = f.$$

For the Riesz logarithmic means we have:

Theorem (Gát, JAT, 2010) Let (n_j) be any convex sequence of natural numbers tending to $+\infty$. Then it holds the almost everywhere relation: $\lim_{N\to\infty}\frac{1}{\log N}\sum_{j=1}^{N}\frac{S_{n_j}f}{j} = f$ for every $f \in L^1(T)$.

Conjecture: The theorem above concerning the Riesz log. means and trig. sys. holds.

Problems:

- What about $(C,1)$ means of $(S_{n_j}f)$ with resp. the Walsh-Kaczmarz system. Nothing is known yet.
- Norm convergence of $(C,1)$ means of $(S_{n_j}f)$ with resp. the Walsh-Paley, Walsh-Kaczmarz system. Not complete characterization.
- Two or more dimensional situation?

Thank you!

Thank you for your attention!