A Note on the Magnitude of Fourier Transform and Walsh-Fourier Transform

Vanda Fülöp

University of Szeged
Bolyai Institute

6th Workshop on Fourier Analysis and Related Fields
Pécs, Hungary, 24-31 August 2017.

Main topics of the talk:

- BACKGROUND, KNOWN RESULTS.

  Fourier coefficients

  Walsh-Fourier coefficients

- JOINT WORK WITH BHIKHA LILA GHODADRA

  Fourier transforms

  Walsh-Fourier transforms

  Remarks, open questions

Known results, Fourier coefficients

- Let $f : \mathbb{T} \to \mathbb{C}$, where $\mathbb{T} := [0, 2\pi)$, $f(2\pi) := f(0)$. We recall that if $f \in L^1(\mathbb{T})$ then the Fourier coefficients of f are defined by

$$\hat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx} dx, \quad k \in \mathbb{Z} .$$

Known results, Fourier coefficients

- Let $f : \mathbb{T} \to \mathbb{C}$, where $\mathbb{T} := [0, 2\pi)$, $f(2\pi) := f(0)$. We recall that if $f \in L^1(\mathbb{T})$ then the Fourier coefficients of f are defined by

$$\hat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx} dx, \quad k \in \mathbb{Z} \, .$$

- THEOREM. $\quad f \in BV([0, 2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\frac{1}{|n|}\right) \, .$

Known results, Fourier coefficients

- Let $f : \mathbb{T} \to \mathbb{C}$, where $\mathbb{T} := [0, 2\pi)$, $f(2\pi) := f(0)$. We recall that if $f \in L^1(\mathbb{T})$ then the Fourier coefficients of f are defined by

$$\hat{f}(k) := \frac{1}{2\pi} \int_{\mathbb{T}} f(x) e^{-ikx} dx, \quad k \in \mathbb{Z} .$$

- THEOREM. $f \in BV([0, 2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\frac{1}{|n|}\right)$.
- We recall that a function $f : [a, b] \to \mathbb{C}$ is said to be of bounded variation over $[a, b]$, in symbol: $f \in BV([a, b])$, if

$$\sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < \infty ,$$

where the supremum is extended over all finite sequences

$a = x_0 < x_1 < x_2 < ... < x_n = b, \quad n = 1, 2, ...$

Known results, Fourier coefficients

- THEOREM. (R. N. Siddiqi, 1972.)

$$f \in BV^{(p)}([0,2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\frac{1}{|n|^{1/p}}\right) \quad (p \geq 1) \ .$$

Known results, Fourier coefficients

- THEOREM. (R. N. Siddiqi, 1972.)

$$f \in BV^{(p)}([0,2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\frac{1}{|n|^{1/p}}\right) \quad (p \geq 1) \ .$$

- DEFINITION. (N. Wiener, 1924.) $f \in BV^{(p)}([a,b])$, $p \geq 1$, that is f is of p-bounded variation over $[a,b]$, if

$$V_p(f,[a,b]) = \sup_{\{I_k\}} \left\{ \left(\sum_k |f(b_k) - f(a_k)|^p \right)^{1/p} \right\} < \infty \ ,$$

in which $\{I_k = [a_k, b_k]\}$ is a sequence of non-overlapping subintervals of $[a,b]$, $V_p(f,[a,b])$ called the p-variation of f on $[a,b]$.

Known results, Fourier coefficients

- THEOREM. (R. N. Siddiqi, 1972.)

$$f \in BV^{(p)}([0,2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\frac{1}{|n|^{1/p}}\right) \quad (p \geq 1) \, .$$

- DEFINITION. (N. Wiener, 1924.) $f \in BV^{(p)}([a,b])$, $p \geq 1$, that is f is of p-bounded variation over $[a,b]$, if

$$V_p(f,[a,b]) = \sup_{\{I_k\}} \left\{ \left(\sum_k |f(b_k) - f(a_k)|^p \right)^{1/p} \right\} < \infty \, ,$$

in which $\{I_k = [a_k, b_k]\}$ is a sequence of non-overlapping subintervals of $[a,b]$, $V_p(f,[a,b])$ called the p-variation of f on $[a,b]$.

- $$p = 1 \; : \quad BV^{(p)} = BV$$

Known results, Fourier coefficients

- DEFINITION. (L. C. Young, 1937.) $f \in \phi BV([a,b])$, that is f is of ϕ-bounded variation over $[a,b]$, if

$$V_\phi(f,[a,b]) = \sup_{\{I_k\}} \left\{ \sum_k \phi(|f(b_k) - f(a_k)|) \right\} < \infty \ ,$$

in which $\{I_k = [a_k, b_k]\}$ is a sequence of non-overlapping subintervals of $[a,b]$ and $\phi : [0,\infty) \to \mathbb{R}$, is a strictly increasing function.

Known results, Fourier coefficients

- DEFINITION. (L. C. Young, 1937.) $f \in \phi BV([a,b])$, that is f is of ϕ-bounded variation over $[a,b]$, if

$$V_\phi(f,[a,b]) = \sup_{\{I_k\}} \left\{ \sum_k \phi(|f(b_k) - f(a_k)|) \right\} < \infty ,$$

  in which $\{I_k = [a_k, b_k]\}$ is a sequence of non-overlapping subintervals of $[a,b]$ and $\phi : [0,\infty) \to \mathbb{R}$, is a strictly increasing function.

-

$$\phi(u) = u \quad : \quad \phi BV = BV$$

$$\phi(u) = u^p \quad : \quad \phi BV = BV^{(p)}$$

Known results, Fourier coefficients

- DEFINITION. (L. C. Young, 1937.) $f \in \phi BV([a,b])$, that is f is of ϕ-bounded variation over $[a,b]$, if

$$V_\phi(f,[a,b]) = \sup_{\{I_k\}} \left\{ \sum_k \phi(|f(b_k) - f(a_k)|) \right\} < \infty \ ,$$

  in which $\{I_k = [a_k, b_k]\}$ is a sequence of non-overlapping subintervals of $[a,b]$ and $\phi : [0,\infty) \to \mathbb{R}$, is a strictly increasing function.

- $$\phi(u) = u \ : \quad \phi BV = BV$$
$$\phi(u) = u^p \ : \quad \phi BV = BV^{(p)}$$

- THEOREM. (M. Shramm, D. Waterman, 1982.)

$$f \in \phi BV([0,2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\phi^{-1}\left(\frac{1}{|n|}\right)\right) \ .$$

- DEFINITION. (D. Waterman, 1972.) $f \in \Lambda BV([a, b])$, that is f is of Λ-bounded variation over $[a, b]$, if

$$V_\Lambda(f, [a, b]) = \sup_{\{I_k\}} \left\{ \sum_k \frac{|f(b_k) - f(a_k)|}{\lambda_k} \right\} < \infty \ ,$$

- DEFINITION. (D. Waterman, 1972.) $f \in \Lambda BV([a,b])$, that is f is of Λ-bounded variation over $[a,b]$, if

$$V_\Lambda(f,[a,b]) = \sup_{\{I_k\}} \left\{ \sum_k \frac{|f(b_k) - f(a_k)|}{\lambda_k} \right\} < \infty \ ,$$

- DEFINITION. (M. Shiba, 1980.) $f \in \Lambda BV^{(p)}([a,b])$, $p \geq 1$, that is f is of p-Λ-bounded variation over $[a,b]$, if

$$V_{p\Lambda}(f,[a,b]) = \sup_{\{I_k\}} \left\{ \left(\sum_k \frac{|f(b_k) - f(a_k)|^p}{\lambda_k} \right)^{1/p} \right\} < \infty \ ,$$

  where $\{I_k = [a_k, b_k]\}$ is just like above, $\{\lambda_k\}$ is a sequence of nondecreasing positive real numbers such that $\sum_{k=1}^{\infty} \frac{1}{\lambda_k}$ diverges.

- DEFINITION. (D. Waterman, 1972.) $f \in \Lambda BV([a,b])$, that is f is of Λ-bounded variation over $[a,b]$, if

$$V_\Lambda(f,[a,b]) = \sup_{\{I_k\}} \left\{ \sum_k \frac{|f(b_k)-f(a_k)|}{\lambda_k} \right\} < \infty \ ,$$

- DEFINITION. (M. Shiba, 1980.) $f \in \Lambda BV^{(p)}([a,b])$, $p \geq 1$, that is f is of p-Λ-bounded variation over $[a,b]$, if

$$V_{p\Lambda}(f,[a,b]) = \sup_{\{I_k\}} \left\{ \left(\sum_k \frac{|f(b_k)-f(a_k)|^p}{\lambda_k} \right)^{1/p} \right\} < \infty \ ,$$

  where $\{I_k = [a_k, b_k]\}$ is just like above, $\{\lambda_k\}$ is a sequence of nondecreasing positive real numbers such that $\sum_{k=1}^{\infty} \frac{1}{\lambda_k}$ diverges.
- $p = 1 \ : \quad \Lambda BV^{(p)} = \Lambda BV \ , \qquad \lambda_k = 1 \ : \quad \Lambda BV^{(p)} = BV^{(p)}$
  $\lambda_k = 1$ and $p = 1 \ : \quad \Lambda BV^{(p)} = BV$

- DEFINITION. (D. Waterman, 1972.) $f \in \Lambda BV([a,b])$, that is f is of Λ-bounded variation over $[a,b]$, if

$$V_\Lambda(f,[a,b]) = \sup_{\{I_k\}} \left\{ \sum_k \frac{|f(b_k) - f(a_k)|}{\lambda_k} \right\} < \infty \ ,$$

- DEFINITION. (M. Shiba, 1980.) $f \in \Lambda BV^{(p)}([a,b])$, $p \geq 1$, that is f is of p-Λ-bounded variation over $[a,b]$, if

$$V_{p\Lambda}(f,[a,b]) = \sup_{\{I_k\}} \left\{ \left(\sum_k \frac{|f(b_k) - f(a_k)|^p}{\lambda_k} \right)^{1/p} \right\} < \infty \ ,$$

  where $\{I_k = [a_k, b_k]\}$ is just like above, $\{\lambda_k\}$ is a sequence of nondecreasing positive real numbers such that $\sum_{k=1}^{\infty} \frac{1}{\lambda_k}$ diverges.
- $p = 1 \ : \quad \Lambda BV^{(p)} = \Lambda BV \ , \qquad \lambda_k = 1 \ : \quad \Lambda BV^{(p)} = BV^{(p)}$

  $\lambda_k = 1$ and $p = 1 \ : \quad \Lambda BV^{(p)} = BV$
- $\lambda_k = k$ and $p = 1 \ : \quad \Lambda BV^{(p)} = HBV$

Known results, Fourier coefficients

- THEOREM. (M. Shramm, D. Waterman, 1982.)

$$f \in \Lambda BV([0,2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(1 \Big/ \sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right) .$$

$$f \in \Lambda BV^{(p)}([0,2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(1 \Big/ \left(\sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right)^{1/p}\right) .$$

Known results, Fourier coefficients

- THEOREM. (M. Shramm, D. Waterman, 1982.)

$$f \in \Lambda BV([0,2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(1 \Big/ \sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right) .$$

$$f \in \Lambda BV^{(p)}([0,2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(1 \Big/ \left(\sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right)^{1/p}\right) .$$

- REMARK. The first estimate is the best possible:

$$\Gamma BV \supseteq \Lambda BV \text{ properly} : \ \exists f \in \Gamma BV : \hat{f}(n) \neq O\left(1 \Big/ \sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right) .$$

Known results, Fourier coefficients

- DEFINITION. (M. Shramm, D. Waterman, 1982.) $f \in \phi\Lambda BV([a,b])$, that is f is of ϕ-Λ-bounded variation over $[a,b]$, if

$$V_{\phi\Lambda}(f,[a,b]) = \sup_{\{I_k\}} \left\{ \sum_k \frac{\phi(|f(b_k) - f(a_k)|)}{\lambda_k} \right\} < \infty$$

in which $\{I_k = [a_k, b_k]\}$, ϕ and $\{\lambda_k\}$ are as before.

Known results, Fourier coefficients

- DEFINITION. (M. Shramm, D. Waterman, 1982.) $f \in \phi\Lambda BV([a,b])$, that is f is of ϕ-Λ-bounded variation over $[a,b]$, if

$$V_{\phi\Lambda}(f,[a,b]) = \sup_{\{I_k\}} \left\{ \sum_k \frac{\phi(|f(b_k)-f(a_k)|)}{\lambda_k} \right\} < \infty$$

in which $\{I_k = [a_k, b_k]\}$, ϕ and $\{\lambda_k\}$ are as before.

- $$\phi(u) = u^p \quad : \quad \phi\Lambda BV = \Lambda BV^{(p)}$$

Known results, Fourier coefficients

- DEFINITION. (M. Shramm, D. Waterman, 1982.) $f \in \phi\Lambda BV([a,b])$, that is f is of ϕ-Λ-bounded variation over $[a,b]$, if

$$V_{\phi\Lambda}(f,[a,b]) = \sup_{\{I_k\}} \left\{ \sum_k \frac{\phi(|f(b_k)-f(a_k)|)}{\lambda_k} \right\} < \infty$$

in which $\{I_k = [a_k, b_k]\}$, ϕ and $\{\lambda_k\}$ are as before.

- $$\phi(u) = u^p \quad : \quad \phi\Lambda BV = \Lambda BV^{(p)}$$

- THEOREM. (M. Shramm, D. Waterman, 1982.)

$$f \in \phi\Lambda BV([0,2\pi]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\phi^{-1}\left(1 \Big/ \sum_{j=1}^{|n|} \frac{1}{\lambda_j}\right)\right) .$$

Known results, Walsh-Fourier coefficients

- We consider the Walsh orthonormal system $\{w_m(x) : m = 0, 1, 2, ...\}$, defined on the unit interval $\mathbb{I} := [0, 1)$.

Known results, Walsh-Fourier coefficients

- We consider the Walsh orthonormal system $\{w_m(x) : m = 0, 1, 2, ...\}$, defined on the unit interval $\mathbb{I} := [0, 1)$.
- Let f be a 1-periodic function in $L^1([0, 1])$, then the Walsh-Fourier coefficients of f defined by:

$$\hat{f}(n) = \int_0^1 f(x) w_n(x)\, dx$$

Known results, Walsh-Fourier coefficients

- We consider the Walsh orthonormal system $\{w_m(x) : m = 0, 1, 2, ...\}$, defined on the unit interval $\mathbb{I} := [0, 1)$.
- Let f be a 1-periodic function in $L^1([0, 1])$, then the Walsh-Fourier coefficients of f defined by:

$$\hat{f}(n) = \int_0^1 f(x) w_n(x)\, dx$$

- THEOREM. (N. J. Fine, 1949.) Let $f \in L^1([0, 1])$.

$$f \in BV([0, 1]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\frac{1}{n}\right) .$$

- THEOREM. (B. L. Ghodadra and J. R. Patadia, 2008.)

$$f \in BV^{(p)}([0,1]) \quad \Rightarrow \quad \hat{f}(n) = O(1/n^{1/p}) \, .$$

$$f \in \phi BV([0,1]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\phi^{-1}(1/n)\right) \, .$$

$$f \in \Lambda BV^{(p)}([0,1]) \quad \Rightarrow \quad \hat{f}(n) = O\left(1 \Bigg/ \left(\sum_{j=1}^{n} \frac{1}{\lambda_j}\right)^{1/p}\right) \, .$$

$$f \in \phi\Lambda BV([0,1]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\phi^{-1}\left(1 \Bigg/ \sum_{j=1}^{n} \frac{1}{\lambda_j}\right)\right) \, .$$

- THEOREM. (B. L. Ghodadra and J. R. Patadia, 2008.)

$$f \in BV^{(p)}([0,1]) \quad \Rightarrow \quad \hat{f}(n) = O(1/n^{1/p}) \; .$$

$$f \in \phi BV([0,1]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\phi^{-1}(1/n)\right) \; .$$

$$f \in \Lambda BV^{(p)}([0,1]) \quad \Rightarrow \quad \hat{f}(n) = O\left(1 \Big/ \left(\sum_{j=1}^{n} \frac{1}{\lambda_j}\right)^{1/p}\right) \; .$$

$$f \in \phi\Lambda BV([0,1]) \quad \Rightarrow \quad \hat{f}(n) = O\left(\phi^{-1}\left(1 \Big/ \sum_{j=1}^{n} \frac{1}{\lambda_j}\right)\right) \; .$$

- REMARK. The 3rd estimate above with $p = 1$ is the best possible:

$$\Gamma BV \supseteq \Lambda BV \text{ properly} : \; \exists f \in \Gamma BV : \hat{f}(n) \neq O\left(1 \Big/ \sum_{j=1}^{n} \frac{1}{\lambda_j}\right) \; .$$

- THEOREM. (Móricz, Fülöp, 2004.) Let $f \in L^1(\mathbb{T}^2)$.

$$f \in BV_V([0,2\pi]^2) \quad \Rightarrow \quad |\hat{f}(k,l)| \le \frac{V(f,[0,2\pi]^2)}{(2\pi)^2 kl} \quad (k,l \neq 0)\ .$$

- THEOREM. (Móricz, Fülöp, 2004.) Let $f \in L^1(\mathbb{T}^2)$.

$$f \in BV_V([0,2\pi]^2) \quad \Rightarrow \quad |\hat{f}(k,l)| \leq \frac{V(f,[0,2\pi]^2)}{(2\pi)^2 kl} \quad (k,l \neq 0) \ .$$

$$\hat{f}(k,l) := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} f(x,y) e^{-i(kx+ly)} dxdy, \quad k,l \in \mathbb{Z} \ .$$

- THEOREM. (Móricz, Fülöp, 2004.) Let $f \in L^1(\mathbb{T}^2)$.

$$f \in BV_V([0,2\pi]^2) \quad \Rightarrow \quad |\hat{f}(k,l)| \le \frac{V(f,[0,2\pi]^2)}{(2\pi)^2 kl} \quad (k,l \neq 0) \ .$$

$$\hat{f}(k,l) := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} f(x,y) e^{-i(kx+ly)} dxdy, \quad k,l \in \mathbb{Z} \ .$$

- Let $R := [a,b] \times [c,d]$. A function $f : R \to \mathbb{C}$ is said to be of bounded variation over R in the sense of Vitali, in symbol: $f \in BV_V(R)$, if

$$\sup \sum_{k=1}^{m} \sum_{l=1}^{n} |f(x_k,y_l) - f(x_{k-1},y_l) - f(x_k,y_{l-1}) + f(x_{k-1},y_{l-1})| < \infty \ ,$$

where the supremum is extended over all finite sequences $a = x_0 < x_1 < ... < x_m = b$ and $c = y_0 < y_1 < ... < y_n = d \quad m,n = 1,2,...$. The supremum denoted by $V(f) = V(f,R)$, is called the total variation of f over the rectangle R.

- THEOREM. (Móricz, Fülöp, 2004.) Let $f \in L^1(\mathbb{T}^2)$.

$$f \in BV_V([0,2\pi]^2) \quad \Rightarrow \quad |\hat{f}(k,l)| \leq \frac{V(f,[0,2\pi]^2)}{(2\pi)^2 kl} \quad (k,l \neq 0) \ .$$

$$\hat{f}(k,l) := \frac{1}{4\pi^2} \int_{\mathbb{T}^2} f(x,y) e^{-i(kx+ly)} dxdy, \quad k,l \in \mathbb{Z} \ .$$

- Let $R := [a,b] \times [c,d]$. A function $f : R \to \mathbb{C}$ is said to be of bounded variation over R in the sense of Vitali, in symbol: $f \in BV_V(R)$, if

$$\sup \sum_{k=1}^{m} \sum_{l=1}^{n} |f(x_k,y_l) - f(x_{k-1},y_l) - f(x_k,y_{l-1}) + f(x_{k-1},y_{l-1})| < \infty \ ,$$

where the supremum is extended over all finite sequences $a = x_0 < x_1 < ... < x_m = b$ and $c = y_0 < y_1 < ... < y_n = d \quad m,n = 1,2,...$. The supremum denoted by $V(f) = V(f,R)$, is called the total variation of f over the rectangle R.

- REMARK. The estimate is exact.

Joint work with B. L. Ghodadra

- Joint work with B. L. Ghodadra

Bhikha Lila Ghodadra,
Associate Professor
Department of Mathematics, Faculty of Science,
The Maharaja Sayajirao University of Baroda,
Vadodara – 390 002 (Gujarat), India

- If $f \in L^1(\mathbb{R})$ then the Fourier transform of f is defined by

$$\hat{f}(t) := \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-itx} dx, \quad t \in \mathbb{R} \ .$$

- If $f \in L^1(\mathbb{R})$ then the Fourier transform of f is defined by

$$\hat{f}(t) := \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-itx} dx, \quad t \in \mathbb{R} .$$

- THEOREM. (Ghodadra, Fülöp, 2015.) Let $f \in L^1(\mathbb{R})$.

$$f \in BV(\mathbb{R}) \quad \Rightarrow \quad \hat{f}(t) = O\left(\frac{1}{|t|}\right), \quad (t \neq 0) .$$

- If $f \in L^1(\mathbb{R})$ then the Fourier transform of f is defined by

$$\hat{f}(t) := \frac{1}{2\pi} \int_{\mathbb{R}} f(x) e^{-itx} dx, \quad t \in \mathbb{R} .$$

- THEOREM. (Ghodadra, Fülöp, 2015.) Let $f \in L^1(\mathbb{R})$.

$$f \in BV(\mathbb{R}) \quad \Rightarrow \quad \hat{f}(t) = O\left(\frac{1}{|t|}\right), \quad (t \neq 0) .$$

- We recall that a function $f : \mathbb{R} \to \mathbb{C}$ is said to be of bounded variation over $\mathbb{R}$, in symbol: $f \in BV(\mathbb{R})$, if

$$\sup \sum_{k=1}^{n} |f(x_k) - f(x_{k-1})| < \infty ,$$

where the supremum is extended over all finite sequences
$-\infty < x_0 < x_1 < x_2 < \ldots < x_n < \infty , \quad n = 1, 2, \ldots .$

- THEOREM. (Ghodadra, Fülöp, 2015.) Let $f \in L^1(\mathbb{R}^2)$.

$$f \in BV_V(\mathbb{R}^2) \quad \Rightarrow \quad \hat{f}(t,s) = O\left(\frac{1}{|ts|}\right) \quad (ts \neq 0)\ .$$

- THEOREM. (Ghodadra, Fülöp, 2015.) Let $f \in L^1(\mathbb{R}^2)$.

$$f \in BV_V(\mathbb{R}^2) \quad \Rightarrow \quad \hat{f}(t,s) = O\left(\frac{1}{|ts|}\right) \quad (ts \neq 0) \ .$$

- $$\hat{f}(t,s) := \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(x,y) e^{-i(tx+sy)} dxdy, \quad t,s \in \mathbb{R} \ .$$

- THEOREM. (Ghodadra, Fülöp, 2015.) Let $f \in L^1(\mathbb{R}^2)$.

$$f \in BV_V(\mathbb{R}^2) \quad \Rightarrow \quad \hat{f}(t,s) = O\left(\frac{1}{|ts|}\right) \quad (ts \neq 0) \ .$$

- $$\hat{f}(t,s) := \frac{1}{4\pi^2} \int_{\mathbb{R}^2} f(x,y) e^{-i(tx+sy)} dxdy, \quad t, s \in \mathbb{R} \ .$$

- A function $f : \mathbb{R}^2 \to \mathbb{C}$ is said to be of bounded variation over the real plane $\mathbb{R}^2$ in the sense of Vitali, in symbol: $f \in BV_V(\mathbb{R}^2)$, if

$$\sup \sum_{k=1}^{m} \sum_{l=1}^{n} |f(x_k, y_l) - f(x_{k-1}, y_l) - f(x_k, y_{l-1}) + f(x_{k-1}, y_{l-1})| < \infty \ ,$$

where the supremum is extended over all finite sequences

$-\infty < x_0 < x_1 < ... < x_m < \infty$ and $-\infty < y_0 < y_1 < ... < y_n < \infty \quad m, n = 1, 2, ...$.

Fourier transforms

- PROBLEM. Estimate $\hat{f}(t,0)$, $t \neq 0$ in terms of $|t|$ (respectively, $\hat{f}(0,s)$, $s \neq 0$ in terms of $|s|$).

Fourier transforms

- PROBLEM. Estimate $\hat{f}(t,0)$, $t \neq 0$ in terms of $|t|$ (respectively, $\hat{f}(0,s)$, $s \neq 0$ in terms of $|s|$).
- PROBLEM. In

$$f \in BV_V(\mathbb{R}^2) \quad \Rightarrow \quad |\hat{f}(t,s)| \leq \frac{V(f,\mathbb{R}^2)}{|ts|} \quad \Rightarrow \ \hat{f}(t,s) = O\left(\frac{1}{|ts|}\right).$$

investigate the accuracy of the constant.

Walsh-Fourier transforms

- We consider the Walsh orthonormal system $\{w_m(x) : m = 0, 1, 2, ...\}$ defined on the unit interval $\mathbb{I} := [0, 1)$.

Walsh-Fourier transforms

- We consider the Walsh orthonormal system $\{w_m(x) : m = 0, 1, 2, ...\}$ defined on the unit interval $\mathbb{I} := [0, 1)$.
- Next, we consider the generalized Walsh functions ψ_y, $y \in \mathbb{R}^+$ and recall following properties:
  - (i) $\psi_k(x) = w_k(x)$ for $k = 0, 1, ..., x \in \mathbb{I}$;
  - (ii) $\psi_y(x \dot{+} t) = \psi_y(x)\psi_y(t)$ for $x, t \in \mathbb{R}^+$ and $x \dot{+} t$ dyadic irrational;
  - (iii) $\psi_y(x) = \psi_x(y)$, $\psi_y(x) = \psi_{[y]}(x)\psi_{[x]}(y)$ for $x, y \in \mathbb{R}^+$;
  - (iv) the functions $\{\psi_j : j = 0, 1, ...\}$ form a complete orthonormal system in each of the intervals of the form $[k, k+1)$, $k = 0, 1, ...$;
  - (v) ψ_j is a periodic extension of w_j from $\mathbb{I}$ to $\mathbb{R}^+$.

Walsh-Fourier transforms

- The Walsh-Fourier transform of an $f \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{f}(y) := \int_0^\infty f(x)\psi_y(x)dx, \quad y \in \mathbb{R}^+.$$

Walsh-Fourier transforms

- The Walsh-Fourier transform of an $f \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{f}(y) := \int_0^\infty f(x)\psi_y(x)dx, \quad y \in \mathbb{R}^+.$$

- The Riemann-Lebesgue lemma holds for Walsh-Fourier transform, that is,

$$\hat{f}(y) \to 0 \text{ as } y \to \infty .$$

Walsh-Fourier transforms

- The Walsh-Fourier transform of an $f \in L^1(\mathbb{R}^+)$ is defined by

$$\hat{f}(y) := \int_0^\infty f(x)\psi_y(x)dx, \quad y \in \mathbb{R}^+.$$

- The Riemann-Lebesgue lemma holds for Walsh-Fourier transform, that is,

$$\hat{f}(y) \to 0 \ \text{ as } \ y \to \infty \ .$$

- THEOREM. Let $f \in L^1(\mathbb{R}^+)$.

$$f \in BV(\mathbb{R}^+) \ \Rightarrow \ \hat{f}(y) = O(1/y), \ \ y \to \infty \ .$$

- THEOREM. Let $f \in L^1((\mathbb{R}^+)^2)$.

$$f \in BV_V((\mathbb{R}^+)^2) \ , \ \xi\eta \neq 0 \quad \Rightarrow \quad \hat{f}(\xi,\eta) = O\left(\frac{1}{\xi\eta}\right), \quad \xi,\eta \to \infty \ .$$

- THEOREM. Let $f \in L^1((\mathbb{R}^+)^2)$.

$$f \in BV_V((\mathbb{R}^+)^2)\ ,\ \xi\eta \neq 0 \quad \Rightarrow \quad \hat{f}(\xi,\eta) = O\left(\frac{1}{\xi\eta}\right), \quad \xi,\eta \to \infty\ .$$

- $$\hat{f}(\xi,\eta) := \int_0^\infty \int_0^\infty f(x,y)\psi_\xi(x)\psi_\eta(y)dxdy, \quad (\xi,\eta) \in (\mathbb{R}^+)^2.$$

- THEOREM. Let $f \in L^1((\mathbb{R}^+)^2)$.

$$f \in BV_V((\mathbb{R}^+)^2)\ ,\ \xi\eta \neq 0 \quad \Rightarrow \quad \hat{f}(\xi,\eta) = O\left(\frac{1}{\xi\eta}\right), \quad \xi,\eta \to \infty\ .$$

- $$\hat{f}(\xi,\eta) := \int_0^\infty \int_0^\infty f(x,y)\psi_\xi(x)\psi_\eta(y)dxdy, \quad (\xi,\eta) \in (\mathbb{R}^+)^2.$$

- PROBLEM. Estimate $\hat{f}(\xi,0)$, $\xi \neq 0$ in terms of ξ (respectively, $\hat{f}(0,\eta)$, $\eta \neq 0$ in terms of η).

- THEOREM. Let $f \in L^1((\mathbb{R}^+)^2)$.

$$f \in BV_V((\mathbb{R}^+)^2)\ ,\ \xi\eta \neq 0 \quad \Rightarrow \quad \hat{f}(\xi,\eta) = O\left(\frac{1}{\xi\eta}\right), \quad \xi,\eta \to \infty\ .$$

- $$\hat{f}(\xi,\eta) := \int_0^\infty \int_0^\infty f(x,y)\psi_\xi(x)\psi_\eta(y)dxdy, \quad (\xi,\eta) \in (\mathbb{R}^+)^2.$$

- PROBLEM. Estimate $\hat{f}(\xi,0)$, $\xi \neq 0$ in terms of ξ (respectively, $\hat{f}(0,\eta)$, $\eta \neq 0$ in terms of η).
- PROBLEM. In

$$f \in BV_V((\mathbb{R}^+)^2) \Rightarrow |\hat{f}(\xi,\eta)| \leq \frac{4^2 V(f,(\mathbb{R}^+)^2)}{\xi\eta} \Rightarrow \hat{f}(\xi,\eta) = O\left(\frac{1}{\xi\eta}\right)$$

investigate the accuracy of the constant.

- THEOREM. Let $f \in L^1((\mathbb{R}^+)^2)$.

$$f \in BV_V((\mathbb{R}^+)^2)\ ,\ \xi\eta \neq 0 \quad \Rightarrow \quad \hat{f}(\xi,\eta) = O\left(\frac{1}{\xi\eta}\right), \quad \xi,\eta \to \infty\ .$$

- $$\hat{f}(\xi,\eta) := \int_0^\infty \int_0^\infty f(x,y)\psi_\xi(x)\psi_\eta(y)dxdy, \quad (\xi,\eta) \in (\mathbb{R}^+)^2.$$

- PROBLEM. Estimate $\hat{f}(\xi,0)$, $\xi \neq 0$ in terms of ξ (respectively, $\hat{f}(0,\eta)$, $\eta \neq 0$ in terms of η).
- PROBLEM. In

$$f \in BV_V((\mathbb{R}^+)^2) \Rightarrow |\hat{f}(\xi,\eta)| \leq \frac{4^2 V(f,(\mathbb{R}^+)^2)}{\xi\eta} \Rightarrow \hat{f}(\xi,\eta) = O\left(\frac{1}{\xi\eta}\right)$$

investigate the accuracy of the constant.
- PROBLEM. Study functions from the classes $BV^{(p)}$, ϕBV, $\Lambda BV^{(p)}$, $\phi\Lambda BV$.

Thank you for your attention.