TRIGONOMETRIC SERIES WITH RANDOM GAPS

Paley and Zygmund, On some series of functions I-III (1930/32)

Series $\sum_{-\infty}^{\infty} \varepsilon_n a_n e^{int}$, $\quad \sum_{-\infty}^{\infty} a_n e^{2\pi i \omega_n} e^{int}$, $\quad \sum_{-\infty}^{\infty} a_n \zeta_n e^{int}$

All are of type $\sum(a_k \cos x + b_k \sin kx)$ with random coefficients

There is no analogous theory for sums $\sum(a_k \cos n_k x + b_k \sin n_k x)$ with random n_k

Motivation:

Sidon (1932) If (n_k) is a B_2 sequence, then the Fourier coefficients of a continuous function can be arbitrarily prescribed on the indices n_k

B_2: all the numbers $n_k + n_\ell$ are different

Erdős (1943) A trigonometric series $\sum(a_k \cos n_k x + b_k \sin n_k x)$ with B_2 frequencies converges almost everywhere under $\sum(a_k^2 + b_k^2) < \infty$ and the sum is in L^4.

Construction of B_2 sequences

There exists a B_2 sequence $n_k \ll k^3$ (greedy algorithm)

Ajtai, Komlós, Szemerédi (1981) $n_k \ll (k/\log k)^3$

Ruzsa (1998) $n_k \ll k^{\sqrt{2}+1+\varepsilon}$

Erdős and Rényi (1960) $\exists n_k \ll k^{2+\varepsilon}$ which is $B_2(r)$ for some r

Central limit theorem for trigonometric sums

Salem and Zygmund (1947) If $n_{k+1}/n_k \geq q > 1$, then

$$\lambda\{x \in (0, 2\pi) : \sum_{k=1}^{N} \sin n_k x < t\sqrt{N}\} \longrightarrow (2\pi)^{1/2} \int_{-\infty}^{t} e^{-u^2/2} du$$

Erdős (1962) Valid if $n_k = e^{c_k \sqrt{k}}$, $c_k \to \infty$, but not for $c_k = c$

Conjecture: CLT holds if $n_k = e^{k^\alpha}$ $(\alpha > 0)$

Kaufman (1980) $n_k = e^{ck^\alpha}$ for almost all $c > 0$

Salem and Zygmund (1954) Let (n_k) be the random sequence obtained by choosing each integer with probability 1/2. Then CLT holds with probability 1 for $\sin n_k x$.

Gap sizes: $n_{k+1} - n_k = O(\log k)$

B (1979) Let n_k be independent random variables, uniformly distributed over disjoint intervals I_k with $|I_k| \to \infty$

Gap sizes: $n_{k+1} - n_k \le \omega_k, \quad \omega_k \to \infty$ arbitrary

Fukuyama (2010) $\exists n_k$ such that $n_{k+1} - n_k = O(1)$ and CLT holds.

Types of random sequences

(a) n_k are independent, $n_k \in I_k$ where I_k are disjoint intervals on $\mathbb{R}$

(b) (Erdős-Rényi) We choose the integer k with probability p_k, independently

(c) (Random walk) $n_{k+1} - n_k$ are i.i.d. random variables, $n_k = \xi_1 + \ldots + \xi_k$

(d) Parametric: m_k is a nonrandom sequence and $n_k = m_k^c$, where c is random

Random walk model (Schatte 1984) $n_k = \xi_1 + \ldots + \xi_k$, (ξ_n) i.i.d. nondegenerate

(a) Convergence and growth of $\sum_{k=1}^N c_k f(n_k x)$, f periodic

(b) Discrepancy of $\{n_k x\}$

Trigonometric series

Theorem (B & Borda & Weber) Let ξ_1 be nondegenerate. Then with probability 1, the series $\sum_{k=1}^\infty c_k e^{2\pi i n_k x}$ converges for almost every x provided $\mathbf{c} \in \ell^2$ and its sum belongs to L^p for any $1 \le p < \infty$. [Paley-Zygmund: $e^{\lambda L^2}$]

Corollary of a.e. convergence $|\sum_{k=1}^N e^{2\pi i n_k x}| = O(\sqrt{N}(\log N)^{1/2+\varepsilon})$ a.e.

Theorem (B & Borda 2017) Let ξ_1 be nondegenerate with characteristic function φ. Then for any $x \neq 0$ we have

$$\limsup_{N\to\infty} \frac{1}{\sqrt{N \log\log N}} \left| \sum_{k=1}^N e^{2\pi i n_k x} \right| = c \frac{\sqrt{1 - |\varphi(2\pi x)|^2}}{|1 - \varphi(2\pi x)|} \quad \text{a.s.}$$

where $c = 1$ or $c = \sqrt{2}$.

General series $\sum_{k=1}^{\infty} c_k f(n_k x)$

Theorem (B & Weber 2009) Let ξ_1 have an **absolutely continuous** distribution and $f \in \mathrm{Lip}(\alpha)$. Then for any $\mathbf{c} \in \ell^2$ the series $\sum_{k=1}^{\infty} c_k f(n_k x)$ converges a.e. with probability 1.

Case of integer n_k: behavior is determined by an interplay between the distribution of the gaps and the rational approximation properties of x

Convergence of $\sum c_k f(kx)$

Carleson (1966) $f(x) = \sin x$: $(c_k) \in \ell^2 \implies$ a.e. convergence

Nikishin (1970) $f(x) = \operatorname{sgn} \sin x$: $(c_k) \in \ell^2$ does not suffice

Aistleitner, B., Seip (2014), Lewko, Radziwill (2015)

$$f \in \mathrm{BV}: \ (c_k) \in \ell^2 (\log \log \ell)^{2+\varepsilon}$$

Gaposhkin (1968), B. (1997) $(c_k) \in \ell^2$ suffices for $f \in \mathrm{Lip}(\alpha)$, $\alpha > 1/2$, but not for $\alpha = 1/2$ Counterexample: $f(x) = \sum_p \pm \frac{\sin 2\pi p x}{p}$

No satisfactory condition for $f \in L^p$ $(p \geq 1)$, $f \in C$

Growth of $\sum_{k=1}^{N} f(kx)$

Khinchin conjecture (1923) If $f \in L^1(T)$, $\int_0^1 f(x)dx = 0$, then

$$\frac{1}{N}\sum_{k=1}^{N} f(kx) = 0 \qquad \text{a.e.}$$

Marstrand (1970) Counterexample for conjecture

Koksma (1953) For $f \sim \sum a_k e^{2\pi ikx}$, $\sum_{k=1}^{\infty} a_k^2 \sigma_{-1}(k) < \infty$ suffices
$\sigma_{-1}(k) = \sum_{d|k} \frac{1}{d} = O(\log\log k)$

Bourgain (1988) New simple counterexample via metric entropy

B & Weber (2014) Koksma's condition is sharp (via metric entropy)

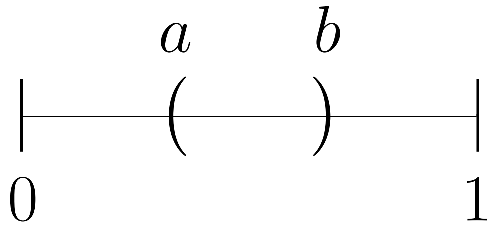
A random result:

Fukuyama (2001) Let $f \in L^1$. Then $\frac{1}{N}\sum_{k=1}^{N} f(k^c x) \to 0$ for almost all $c > 1$ and almost every x

Discrepancy of $\{n_k x\}$

$(x_n) \subset (0,1)$ is uniformly distributed mod 1 if

$$\frac{1}{N}\#\{k \le N : \; x_k \in (a,b)\} \longrightarrow b-a$$



$$D_N(x_1,\ldots,x_N) = \sup_{0\le a<b\le 1} \left| \frac{N(a,b)}{N} - (b-a) \right| =$$

$\swarrow$

$\#$ of terms of $x_1,\ldots,x_N$ in (a,b)

(x_n) is UD $\iff$ $D_N(x_1,\ldots,x_N) \to 0$

Weyl (1916) For any increasing sequence (n_k) of integers, $\{n_k\alpha\}$ is uniformly distributed for almost all α.

Exceptional set is undetermined!

Discrepancy of $\{n_k\alpha\}$?

Two known cases:

Philipp (1975) If $n_{k+1}/n_k \geq q > 1$, then

$$D_N(\{n_k\alpha\}) = O\left(\sqrt{\frac{\log\log N}{N}}\right) \quad \text{and} \quad D_N(\{n_k\alpha\}) = \Omega\left(\sqrt{\frac{\log\log N}{N}}\right) \text{ a.e.}$$

Fukuyama (2008) Precise constant for $n_k = 2^k$: $\frac{\sqrt{42}}{9}$

Ostrowski, Khinchin, Hardy& Littlewood (1920-)

Discrepancy of $\{k\alpha\}$ depends on the **rational approximability** of α

$\|x\|$ = distance of x from nearest integer

Theorem. Assume

$$0 < \liminf_{q\to\infty} q^{\gamma}\|q\alpha\| < \infty \qquad (\gamma \geq 1) \tag{1}$$

Then

$$D_N(\{k\alpha\}) = O(N^{-1/\gamma}), \quad D_N(\{k\alpha\}) = \Omega(N^{-1/\gamma}) \qquad \text{for } \gamma > 1$$

and

$$D_N(\{k\alpha\}) = O\left(\frac{\log N}{N}\right), \quad D_N(\{k\alpha\}) = \Omega\left(\frac{\log N}{N}\right) \qquad \text{for } \gamma = 1 \tag{2}$$

Meaning of (1):

$$\text{lower bound} \iff |\alpha - p/q| \geq Cq^{-\gamma-1} \quad \text{for all fractions } p/q$$

By Schmidt's theorem, the lower bound in (2) holds for the discrepancy of any sequence (x_k)

No precise results beyond these two cases!

$D_N(\{k^2\alpha\}) = O\left(\frac{(\log N)^{5/4}}{\sqrt{N}}\right), \qquad D_N(\{k^2\alpha\}) = \Omega\left(\frac{(\log N)^{1/4}}{\sqrt{N}}\right)$ Hardy & Littlewood, Walfisz

Discrepancy of random sequences $\{n_k\alpha\}$

$n_k = \xi_1 + \ldots + \xi_k$, where $\xi_1, \xi_2, \ldots$ are i.i.d. random variables

Case 1: ξ_1 is **absolutely continuous**

Theorem. (B & Raseta (2015)) For any α we have with probability 1

$$\limsup_{N\to\infty} \sqrt{\frac{N}{2\log\log N}} D_N^*(\{n_k\alpha\}) = \sup_{y\in B_\Gamma} \|y\|_\infty$$

and

$$\limsup_{N\to\infty} \sqrt{\frac{N}{2\log\log N}} D_N^{(p)}(\{n_k\alpha\}) = \sup_{y\in B_\Gamma} \|y\|_p \qquad (p \geq 1)$$

where B_Γ is the unit ball of the reproducing kernel Hilbert space determined by the covariance function

$$\Gamma(s, s', x) = \mathbb{E} g_s(U) g_{s'}(U) + \sum_{j=1}^{\infty} \mathbb{E} g_s(U) g_{s'}(U + n_j x) + \sum_{j=1}^{\infty} \mathbb{E} g_{s'}(U) g_s(U + n_j x).$$

Here $g_s = I_{(0,s)} - s$ is the centered indicator function of the interval $(0, s)$.

Case 2: ξ_1 is integer valued

The discrepancy of $\{n_k\alpha\}$ is determined by a delicate interplay between the distribution of X_1 and the Diophantine properties of α

We assume $0 < \liminf_{q\to\infty} q^\gamma \|q\alpha\| < \infty$ for some $\gamma \geq 1$.

Theorem 1 (B & Borda) For any nondegenerate ξ_1 we have

$$D_N(\{n_k\alpha\}) = \Omega\left(\sqrt{\frac{\log\log N}{N}}\right) \quad \text{a.s.}$$

Theorem 2 (B & Borda) Assume $P(X_1 = 1) = P(X_1 = 2) = 1/2$. Then for $\gamma \leq 2$

$$D_N(\{n_k\alpha\}) = O\left(\sqrt{\frac{\log\log N}{N}}\log N\right), \quad D_N(\{n_k\alpha\}) = \Omega\left(\sqrt{\frac{\log\log N}{N}}\right) \quad \text{a.s.}$$

and for $\gamma > 2$

$$D_N(\{n_k\alpha\}) = O\left(N^{-\frac{1}{\gamma}}(\log\log N)^{\frac{1}{\gamma}}\right), \quad D_N(\{n_k\alpha\}) = \Omega\left(N^{-\frac{1}{\gamma}}\right) \quad \text{a.s.}$$

Critical value: $\gamma = 2$ $\qquad |\alpha - p/q| \leq Cq^{-3}$