The Plancherel – Polya inequality for entire functions of exponential type in $L^2(\mathbb{R})$

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Notation and statement of the problem

Let $\mathfrak{M}_\nu^p$ be the set of entire functions f of exponential type $\leqslant \nu$ with the property $f|_{\mathbb{R}} \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$.
Recall that $f(z)$ has exponential type ν if for any $\varepsilon > 0$ there exists A_ε such that

$$|f(z)| \leqslant A_\varepsilon e^{(\nu+\varepsilon)|z|}, \quad z \in \mathbb{C}.$$

Problem

Find the smallest constant $c_p(\nu)$ in the inequality

$$\sum_{k\in\mathbb{Z}} |f(k)|^p \leqslant c_p(\nu) \|f\|_p^p, \quad f \in \mathfrak{M}_\nu^p.$$

The problem was posed by G.Tamberg from Tallinn University of Technology.
We found a solution for $p = 2$.

Known results

Theorem (Plancherel–Polya, 1937)

For any $f \in \mathfrak{M}_\pi^p$, $1 < p < \infty$, the sequence $\{f(k)\}_{k\in\mathbb{Z}}$ belongs to ℓ_p, and there exist m, $M > 0$ such that:

$$m\left(\sum_{k\in\mathbb{Z}} |f(k)|^p\right)^{1/p} \leqslant \|f\|_p \leqslant M\left(\sum_{k\in\mathbb{Z}} |f(k)|^p\right)^{1/p}.$$

Conversely, for any $\{c_k\}_{k\in\mathbb{Z}} \in \ell_p$, $1 < p < \infty$, the series

$$\sum_{k\in\mathbb{Z}} c_k \frac{\sin(\pi(z-k))}{\pi(z-k)}$$

converges in $L^p(\mathbb{R})$ to $f \in \mathfrak{M}_\pi^p$, f is the unique solution of the interpolation problem $f(k) = c_k$, $k \in \mathbb{Z}$.

Known results

Theorem (Whittaker–Kotel'nikov–Shannon)

For any function $f \in \mathfrak{M}^2_\pi$,

$$f(z) = \sum_{k\in\mathbb{Z}} f(k) \frac{\sin(\pi(z-k))}{\pi(z-k)} \quad \text{and} \quad \sum_{k\in\mathbb{Z}} |f(k)|^2 = \|f\|_2^2.$$

It follows that

$$c_2(\nu) = 1, \quad \nu \leqslant \pi,$$

and the inequality

$$\sum_{k\in\mathbb{Z}} |f(k)|^2 \leqslant c_2(\nu)\|f\|_2^2.$$

turns into equality for any $f \in \mathfrak{M}^2_\pi$.

Known results

$$\sum_{k\in\mathbb{Z}} |f(k)|^p \leqslant c_p(\nu)\|f\|_p^p, \quad f \in \mathfrak{M}_\nu^p.$$

Nikol'skii, 1951

$$c_p(\nu) \leqslant (1+\nu)^p, \quad 1 \leqslant p < \infty.$$

Known results

Donoho, Logan, Norvidas studied more general problem

$$\sum_{k\in\mathbb{Z}} |f(\lambda_k)|^p \leqslant c_p(\nu)\|f\|_p^p, \quad f \in \mathfrak{M}_\nu^p,$$

for uniformly discrete $\{\lambda_k\}$, i.e. $\inf\{|\lambda_\ell - \lambda_k| : \ell \neq k\} > 0$.

For $\lambda_k = k$ and $0 < \nu\delta < \pi$ they results give:

Donoho and Logan, 1992

$$c_2(\nu) \leqslant \frac{2\nu([\delta]+1)}{\nu\delta + \sin(\nu\delta)}, \quad c_1(\nu) \leqslant \frac{\nu([\delta]+1)}{2\sin(\nu\delta/2)}.$$

Norvidas, 2014

$$c_p(\nu) \leqslant \frac{[\delta]+1}{2\delta\|\cos(\nu\delta\cdot)\|_{L^p[0,1/2]}^p}, \quad 1 \leqslant p \leqslant 2.$$

$$c_2(\nu) \leqslant \inf_\delta \frac{2\nu([\delta]+1)}{\nu\delta + \sin(\nu\delta)} = \lim_{\delta\to\pi/\nu-0} \frac{2\nu([\delta]+1)}{\nu\delta + \sin(\nu\delta)} = \frac{2\nu}{\pi}, \quad \nu > \pi.$$

Main result

$$\sum_{k\in\mathbb{Z}} |f(k)|^2 \leqslant c_2(\nu)\|f\|_2^2, \quad f\in\mathfrak{M}_\nu^2. \qquad (1)$$

Theorem

For any $\nu > 0$

$$c_2(\nu) = \left\lceil \frac{\nu}{\pi} \right\rceil .$$

If $f = \widehat{g}$, where g is an even function from $L^2\left[-\frac{\nu}{2\pi}, \frac{\nu}{2\pi}\right]$ and $g(t) = g(t-[t])$, $t\in\left[0, \frac{\nu}{2\pi}\right]$, then inequality (1) turns into equality.

$$\widehat{g}(x) = \int_{\mathbb{R}} g(t)e^{-2\pi i x t}dt$$

Proof. Estimate of $c_2(\nu)$ from above

Let $\nu > \pi$ and $f \in \mathfrak{M}^2_\nu$. Take $m = \lceil \nu/\pi \rceil$, then $\nu/m \leqslant \pi$.
Consider the function

$$\varphi(z) = f(z/m).$$

It is clear, that $\varphi \in \mathfrak{M}^2_\pi$.
Now we can apply the known result for $\mathfrak{M}^2_\pi$ and obtain

$$\sum_{k\in\mathbb{Z}} |f(k/m)|^2 = \sum_{k\in\mathbb{Z}} |\varphi(k)|^2 = \int_{\mathbb{R}} |\varphi(x)|^2 dx = m \int_{\mathbb{R}} |f(x)|^2 dx.$$

Since

$$\sum_{k\in\mathbb{Z}} |f(k)|^2 \leqslant \sum_{k\in\mathbb{Z}} |f(k/m)|^2,$$

we conclude that $c_2(\nu) \leqslant m = \lceil \nu/\pi \rceil$.

Proof. Estimate of $c_2(\nu)$ from below

$$\sum_{k\in\mathbb{Z}} |f(k)|^2 \leqslant c_2(\nu)\|f\|_2^2, \quad f \in \mathfrak{M}_\nu^2 \tag{1}$$

The following problem gives an estimate of $c_2(\nu)$ from below:

$$\sum_{k\in\mathbb{Z}} (g * g)(k) \leqslant \widetilde{c}_2(\nu)\|g\|_2^2, \quad g \in L^2\left[-\frac{\nu}{2\pi}, \frac{\nu}{2\pi}\right]. \tag{2}$$

Theorem

$$c_2(\nu) = \widetilde{c}_2(\nu).$$

If a function g is extremal for (2), then $f = \widehat{g}$ is extremal for (1). Equality in (2) is achieved if and only if g is even and $g(t) = g(t - [t])$, $t \in [0, \frac{\nu}{2\pi}]$.

The last condition means that g is symmetric on any interval $[k, k+1] \subset \left[\frac{-\nu}{2\pi}, \frac{\nu}{2\pi}\right]$, $k \in \mathbb{Z}$.

Thank you for your attention!

References

- E. Stein, G. Weiss. Introduction to Fourier Analysis on Euclidean Spaces. M: Mir, 1974.
- S. M. Nikol'skii. Approximation of Functions of Several Variables and Embedding Theorems. M: Nauka, 1977.
- Levin B. Ya. Lectures on Entire Functions. Translations of mathematical monographs, V olume 150, 1996 by the American mathematical society.
- Norvidas S. Concentration of L^p-bandlimited functions on discrete sets. Lithuanian Mathematical Journal, 54 (4) (2014) 471–481.
- D. L. Donoho, B. F. Logan. Signal recovery and the large sieve. SIAM J. 52 (2) (1992) 577–591.