# Interrelation between integral and uniform approximations

The 6th Workshop on Fourier Analysis and Related Fields, Pécs, Hungary, August 24–31, 2017

A. G. Babenko

Ural Federal University, Ekaterinburg, Russia

Yu. V. Kryakin

Mathematical Institute, Wrocław University, Poland

**J. V. Poncelet (1835)** solved the problem, which reduces to finding a function of the following form:

$$f(x) = \frac{d_0 + d_1 x}{\sqrt{1 + x^2}} - 1$$

that deviates least from zero at a given interval $[a, b]$. That is, he found $\inf_{d_0, d_1} \|f\|_{C[a,b]}$.

Below $r, n \in \mathbb{N}$, $x, t, \xi \in \mathbb{R}$, $z \in \mathbb{C}$;

$\mathcal{P}_n$ — subspace of algebraic polynomials

$P(x) = d_0 + d_1 x + d_2 x^2 + \ldots + d_n x^n$,

$\deg P \leqslant n$.

**P. L. Chebyshev (1859)** found a fraction, least deviating from zero in $C[-1,1]$ among fractions of the form

$$\frac{x^r - P(x)}{Q(x)}, \qquad P \in \mathcal{P}_{r-1},$$

where $Q$ — fixed polynomial, $\deg Q \leqslant r$,

$$Q(x) \neq 0 \quad \text{for all} \quad x \in [-1,1].$$

Let $\mathcal{T}_n$ be a subspace of real-valued trigonometric polynomials

$$\tau(t) = a_0 + \sum_{\nu=1}^{n} (a_\nu \cos \nu t + b_\nu \sin \nu t) = \sum_{\nu=-n}^{n} c_\nu e^{i\nu t},$$

$a_\nu, b_\nu \in \mathbb{R}$; and $\mathbb{T} = [-\pi, \pi)$ — period.

Consider a fraction

$$\frac{\cos rt \; - \; \tau(t)}{\vartheta(t)}, \qquad \tau \in \mathcal{T}_{r-1}, \qquad (1)$$

in which $\vartheta$ is a fixed positive cosine polynomial

of degree at most $r$.

**Chebyshev's**, in fact, found the fraction that **deviates least from zero** in period $[-\pi, \pi)$ among all fractions of the form (1).

Let us formulate several known results in **"trigonometric form",** despite the fact that the results were originally obtained in **"algebraic form".**

Let $w$ **be a positive** cosine polynomial, $\deg w \leqslant 2r$. **A. A. Markov (1884)** found a formula for fraction

$$c(t) := \frac{\cos rt - \tau^*(t)}{\sqrt{w(t)}},$$

**the least deviating from zero** in $C(\mathbb{T})$, among fractions of the form

$$\frac{\cos rt - \tau(t)}{\sqrt{w(t)}}, \qquad \tau \in \mathcal{T}_{r-1}. \qquad (2)$$

He indicated the formula of an extremal

fraction $\quad c(t) = \lambda \cos \varphi(t), \quad t \in [0, \pi], \quad (3)$

with explicit formulas for $\lambda$ and a continuous

function $\varphi(t)$.

The value of $\varphi(t)$ runs a segment of length $r\pi$,

when $t$ runs a segment $[0, \pi]$.

Similar representations for fractions that are extremal in general problems, pointed out **S. N. Bernstein (1912, 1926, 1930).** In particular, for $\xi = 0, -\pi/2$, he found a fraction

$$c_\xi(t) := \frac{\cos(rt+\xi) - \tau^*(t)}{\sqrt{w(t)}}, \qquad \tau^* \in \mathcal{T}_{r-1},$$

that realizes the minimum

$$\min_{\tau\in\mathcal{T}_{r-1}} \int\limits_{-\pi}^{\pi} \Psi\left(\frac{|\cos(rt+\xi)-\tau(t)|}{\sqrt{w(t)}}\right) dt =$$

$$= \int\limits_{-\pi}^{\pi} \Psi\left(|c_\xi(t)|\right) dt. \qquad (4)$$

Here $w \in \mathcal{T}_{2r}$ **is a positive** cosine-polynomial and $\Psi(u)$ is an arbitrary increasing and convex function on $\mathbb{R}_+ = [0, \infty)$.

It turned out that **the extremal fraction does not depend on** $\Psi$.

In particular, for $\xi = 0$ the fraction $c_0(t)$ coincides with $c(t)$ (see (3)).

**Theorem (G. Szegö, 1964)**

*Let* $r \geqslant n, \quad \xi \in \mathbb{R},$

$$p(z) = \prod_{j=1}^{r-n} (z - z_j) \qquad (z_j \in \mathbb{C}, \quad \textit{all} \quad |z_j| < 1).$$

*Then the fraction*

$$G(t) := \frac{\mathrm{Re}\{e^{i\xi} z^{2n-r} p^2(z)\}}{|p^2(z)|}, \quad z = e^{it}, \quad t \in \mathbb{R},$$

*least deviates from zero in the metric of the* *space* $C(\mathbb{T})$ *among all fractions of the form*

$$\frac{\cos(rt+\xi)-\tau(t)}{|p^2(e^{it})|}, \qquad \tau \in \mathcal{T}_{r-1}.$$

Let us compare this Theorem with the results

of Markov and Bernstein (see (2) − (4)).

It is natural to assume that the Szegö Theorem remains valid if in its formulation $p^2(e^{it})$ is replaced by $P(e^{it})$, increasing the degree of the polynomial $P$ by two times as a result of doubling the number of zeros inside the unit disk.

Indeed, the following statement holds.

**Theorem.** *Let* $r \geqslant n, \quad \xi \in \mathbb{R},$

$$P(z) = \prod_{j=1}^{2(r-n)} (z - z_j), \qquad z_j \in \mathbb{C}, \quad \textit{all} \quad |z_j| < 1.$$

*Then the fraction*

$$\frac{\mathrm{Re}\{e^{i\xi} z^{2n-r} P(z)\}}{|P(z)|}, \quad z = e^{it},$$

*least deviates from zero in the metric of the* *space* $C(\mathbb{T})$ *among all fractions of the form*

$$\frac{\cos(rt+\xi)-\tau(t)}{|P(e^{it})|}, \quad \tau \in \mathcal{T}_{r-1}.$$

**F. Peherstorfer (1979)** obtained an integral analog of the Szegö Theorem. We state his result in the form of the following Theorem and Corollary.

**Theorem (F. Peherstorfer, 1979)**

*Let* $r \geqslant n$, $\xi \in \mathbb{R}$, $p(z) = \prod_{j=1}^{r-n} (z - z_j)$

$z_j \in \mathbb{C}$, *all* $|z_j| < 1$. *Then the fraction*

$$G(t) := \frac{\mathrm{Re}\left\{e^{i\xi} z^{2n-r} p^2(z)\right\}}{|p^2(z)|}, \quad z = e^{it},$$

*is unique, deviating least from zero in* $L(\mathbb{T})$

*among the fractions of the form*

$$\frac{\cos(rt+\xi)-\tau(t)}{|p^2(e^{it})|}, \qquad \tau \in \mathcal{T}_{r-1}.$$

*More precisely*

$$\min_{\tau \in \mathcal{T}_{r-1}} \left\| \frac{\cos(rt+\xi)-\tau(t)}{|p^2(e^{it})|} \right\|_L = \|G\|_L = 4.$$

**Corollary.** *Let the conditions of the Peherstorfer Theorem be satisfied. Then for any* $\tau \in \mathcal{T}_{r-1}$ *we have the equality*

$$\int_{\mathbb{T}} \text{sign}\left\{\text{Re}\left[e^{i\xi} z^{2n-r} p^2(z)\right]\right\} \frac{\tau(t)}{|p^2(z)|}\, dt = 0, \qquad z = e^{it}. \tag{5}$$

Substituting polynomial $\tau$ of the form

$$\tau(t) = |p^2(e^{it})|\, f(t), \quad \text{where} \quad f \in \mathcal{T}_{n-1},$$

in (5), we obtain ($z = e^{it}$)

$$\int_{\mathbb{T}} \operatorname{sign}\left\{\mathrm{Re}\left[e^{i\xi} z^{2n-r} p^2(z)\right]\right\} f(t)\, dt = 0 \qquad (6)$$

for all $f \in \mathcal{T}_{n-1}$.

Recall that $|p^2(e^{it})|$ is a positive trigonometric polynomial of degree $r - n$.

The orthogonality relation (6) is half of the following theorem, containing necessary and sufficient conditions for a set of points $\{t_k\}_{k=1}^{2r}$ in $[t_1, t_1 + 2\pi)$ to be canonical for $\mathcal{T}_{n-1}$.

**Theorem**

**(Ya. Geronimus, F. Peherstorfer)**

*Let* $r \geqslant n$,

$$g(t) = a_0 + \sum_{k=1}^{r-1} (a_k \cos kt + b_k \sin kt) + \cos(rt + \xi).$$

**Then the conjunction of the following two conditions**:

(1) *the number of changes of the sign of the polynomial* $g$ *on* $\mathbb{T}$ *is* $2r$*,*

(2) *the set of zeros* $t_1 < t_2 < \cdots < t_{2r}$ *of the polynomial* $g$ *on* $[t_1, t_1 + 2\pi)$ *is canonical for* $\mathcal{T}_{n-1}$

**is equivalent to condition**

(3) *there exists a polynomial*

$$p(z) = \prod_{j=1}^{r-n} (z - z_j), \quad \textit{all} \;\; z_j \in \mathbb{C}, \;\; |z_j| < 1,$$

*such that*

$$g(t) = \mathrm{Re}\left\{e^{i\xi} z^{2n-r} p^2(z)\right\}, \quad z = e^{it}, \quad t \in \mathbb{T}.$$

This Theorem was proved by

Ya. L. Geronimus (1939) for $n \leqslant r < 3n$,

and F. Peherstorfer (1979) in the general case.

**Remark.** Let $p(z) = \prod_{j=1}^{r-n} (z - z_j)$, all $|z_j| < 1$. Then not only the set of zeros of the trigonometric polynomial

$$g(t) = \mathrm{Re}\left\{e^{i\xi} z^{2n-r} p^2(z)\right\}, \quad z = e^{it}, \quad t \in \mathbb{T},$$

is canonical for $\mathcal{T}_{n-1}$, but the set of zeros of the trigonometric polynomial

$$\tilde{g}(t) = \operatorname{Im}\left\{e^{i\xi} z^{2n-r} p^2(z)\right\}, \qquad z = e^{it},$$

is canonical for $\mathcal{T}_{n-1}$,

since $\operatorname{Im}\omega = \operatorname{Re}(-i\omega)$ for any $\omega \in \mathbb{C}$.

Thus, we have that the zeros of the fraction

$G(t)$ on $\mathbb{T}$ form a canonical set for $\mathcal{T}_{n-1}$.

G. Szegö (1964) obtained the formula

$$G(t) = \cos\varphi(t),$$

where $\varphi(t)$ is a strictly increasing continuously differentiable function whose value ranges over a half-interval of length $2\pi r$, when $t$ runs $\mathbb{T}$.

Hence the Remark becomes clear why the collection of zeros of the derivative of $G'(t)$ on $\mathbb{T}$ (alternance) is the canonical collection for $\mathcal{T}_{n-1}$. Really,

$$G'(t) = -\varphi'(t) \sin \varphi(t) =$$

$$= -\varphi'(t) \frac{\text{Im}\left\{e^{i\xi} z^{2n-r} p^2(z)\right\}}{|p(z)|^2}, \quad z = e^{it}.$$

Let $h \in (0, \pi]$,

$$\mathbf{1}_{(-h,h)}(t) = \begin{cases} 1, & t \in (-\pi, \pi), \\ 0, & h \leqslant |t| \leqslant \pi, \end{cases}$$

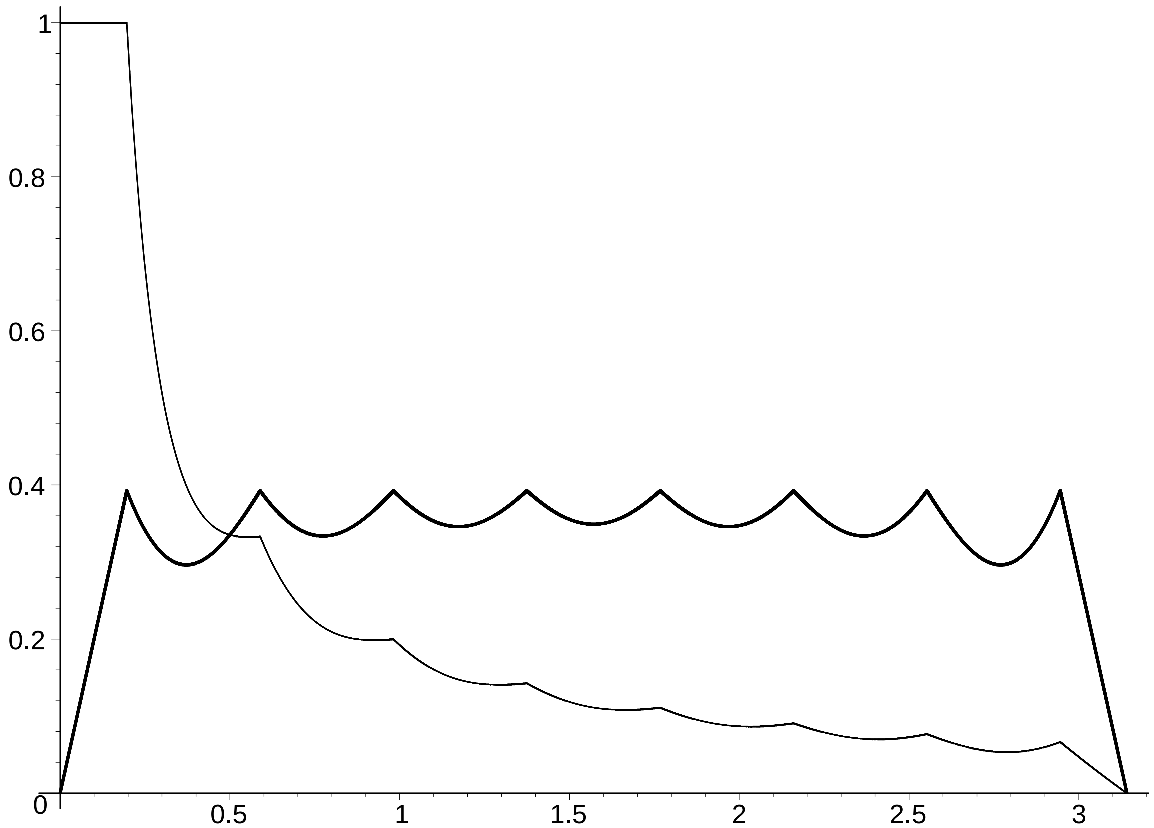
is the characteristic function of the interval

$(-h, h)$, periodically extended to $\mathbb{R}$

with period $2\pi$.

For $h \in (0, \pi]$, we set

$$\mathfrak{I}_n(h) := E_{n-1}\big(\mathbf{1}_{(-h,h)}\big)_L.$$



1
0.8
0.6
0.4
0.2
0
0.5
1
1.5
2
2.5
3

**Theorem.** *Let* $f \in C(\mathbb{T})$, $n \in \mathbb{N}$, $\pi/(2n) < h \leqslant \pi$. *Then the inequalities*

$$E_{n-1}(f) \leqslant \frac{1}{2h - \mathfrak{I}_n(h)} \int_0^h \omega_2(f,t)\,dt \leqslant \frac{\omega_2(f,h)}{2 - \mathfrak{I}_n(h)h^{-1}}$$

*hold.* *Here*

$$E_{n-1}(f) = E_{n-1}(f)_C, \qquad \omega_2(f,t) = \omega_2(f,t)_C.$$

*As a consequence, we have the inequality*

$$E_{n-1}(f) \leqslant \frac{\omega_2(f,h)}{2-\pi/(nh)}.$$

THANK YOU FOR ATTENTION!

KOSZONOM

A FIGYELMET!